Process Control
Fourth Class

Teacher:Suha Mahdi Saleh

Reference
Process systems analysis and control By Coughanowr,D.R.

Control System Objectives

- Economic Incentive
- Safety
- Equipment Protection
- Reduce variability
- Increase efficiency
- Ensure the stability of a process
- Elimination of routine

Definitions: System: It is a combination of components that act together and perform a certain objective.

Plant: It is the machine of which a particular quantity or condition is to be controlled.

Process: Is defined as the changing or refining of raw materials that pass through or remain in a liquid, gaseous, or slurry state to create end products.

Control: In process industries refers to the regulation of all aspects of the process. Precise control of level, pH, oxygen, foam, nutrient, temperature, pressure and flow is important in many process applications. **Sensor**: A measuring instrument, the most common measurements are of flow (F), temperature (T), pressure (P), level (L), pH and composition (A, for analyzer). The sensor will detect the value of the measured variable as a function of time.

Set point: The value at which the controlled parameter is to be maintained.

Controller: A device which receives a measurement of the process variable, compares with a set point representing the desired control point, and adjusts its output to minimize the error between the measurement and the set point.

Error Signal: The signal resulting from the difference between the set point reference signal and the process variable feedback signal in a controller.

Feedback Control: A type of control whereby the controller receives a feedback signal representing the condition of the controlled process variable, compares it to the set point, and adjusts the controller output accordingly.

Steady-State: The condition when all process properties are constant with time, transient responses having died out.

Transmitter: A device that converts a process measurement (pressure, flow, level, temperature, etc.) into an electrical or pneumatic signal suitable for use by an indicating or control system.

Controlled variable: Process output which is to be maintained at a desired value by adjustment of a process input.

Manipulated variable: Process input which is adjusted to maintain the controlled output at set point.

Disturbance: A process input (other than the manipulated parameter) which affects the controlled parameter.

Process Time Constant(τ): Amount of time counted from the moment the variable starts to respond that it takes the process variable to reach 63.2% of its total change.

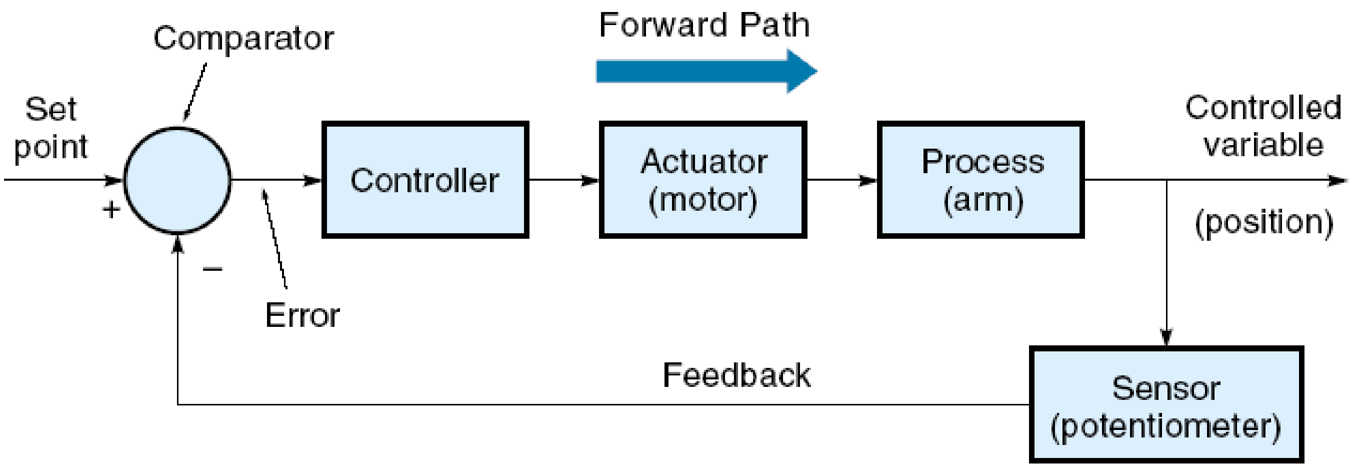
Block diagram: It is relationship between the input and the output of the system. It is easier to visualize the control system in terms of a block diagram.

Transfer Function: it is the ratio of the Laplace transform of output (response function) to the Laplace transform of the input (driving force) under assumption that all initial conditions are zero unless that given another value. e.g. the transfer function of the above block diagram is $G(s) = Y(s)/X(s)$

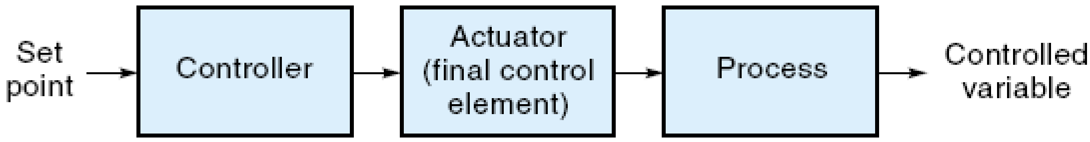
Closed-loop control system: It is a feedback control system which the output signalshas a direct effect upon the control action

Advantage: more accurate than the open-loop control system.

Disadvantages: (1) Complex and expensive (2) The stability is the major problem in closed-loop controlsystem



Open-loop control system: It is a control system in which the output has no effect upon the control action. (The output is neither measured nor fed back for comparison with the input).



Advantages: (1) Simple construction and ease of maintenance. (2) Less expensive than closed-loop control system. (3) There is no stability problem.

Disadvantages: (1) Disturbance and change in calibration cause errors; and output may be different from what is desired. (2) To maintain the required quality in the output, recalibration is necessary from time to time ***Note:*** any control system which operates on a time basis is open-loop control system, e.g. washing machine, traffic light …etc.

The transfer function: The dynamic behavior of the system is described by transfer function (T.F) T.F=Laplace transform of the output (response)/Laplace transform of the input (forcing function disturbance)

T.F=G(s)=y(s)/x(s) This definition is applied to linear systems

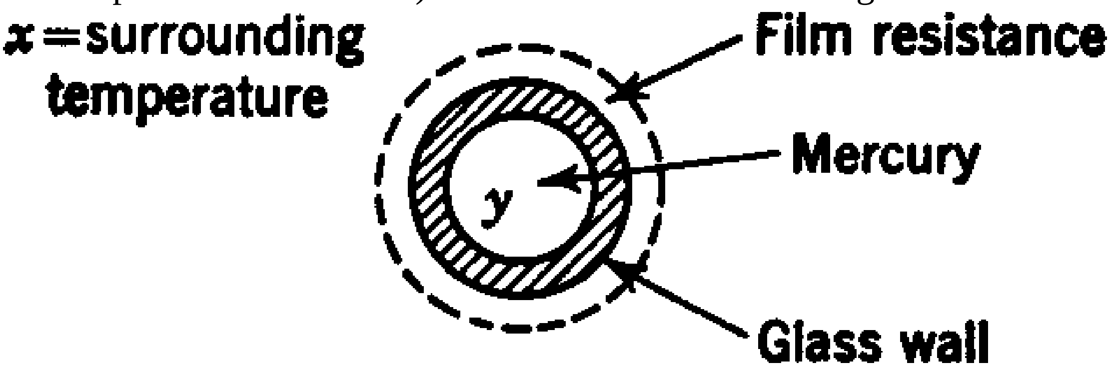
Development of T.F for first order system:

Mercury Thermometer: It is a measuring device used to measure the temperature of a stream.Consider a mercury in glass thermometer to be located in a flowing stream of fluid for which the temperature x varies with time. The object is to calculate the time variation of the thermometer reading y for a particular change of x

The following assumptions will be used in this analysis:-

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e., the resistance offered by the glass and mercury is neglected).
2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.
3. The glass wall containing the mercury does not expand or contract during the transient response.

It is assumed that the thermometer is initially at steady state. This means that, before time zero, there is no change in temperature with time. At time zero the thermometer will be subjected to some change in the surrounding temperature x(t). (i.e at t<0 , x(t)= y(t) =constant there is no change in temperature with time). At t=0 there is a change in the surrounding temperature x(t)



Unsteady state energy balance:

Input-output=accumulation

Output=0

Input=hA(x-y)

Accumulation=d/dt mcp(y-y_{ref})

hA(x-y)= d/dt mcp(y-y_{ref})

hA(x-y)= dy/dt mcp …….1

$y_{ref} = 0$

1st order differential equation

Where

A: area of the bulb

Cp: heat capacity of mercury

m: mass of mercury in the bulb

t: time

h: film heat transfer coefficient h depend on the flow rate and properties of the surrounding fluid and the dimension of the bulb. The dynamic behavior must be defined by a deviation variables.

At steady state (s.s.) , t<0 , x(t)=constant=x_s ,

y(t)=constant=y_s ,

x(t)=constant=x_s

accumulation = m Cp dy/dt=0 at steady state

hA$(x_s - y_s)$=0…….2

eq 1-eq 2

hA(x-x_s)-hA(y-y_s)= dy/dt mcp

x-x_s=X, y-y_s=Y differentiate dy/dt=dY/dt

hAX-hAY=mcp dY/dt take laplace transformation

hAX(s)-hAY(s)=mcp s Y(s)

hAX(s)= Y(s)(mcps+hA)

$$G(s)=\frac{output}{input}=\frac{Y(s)}{X(s)}=\frac{hA}{hA+mcps}=\frac{1}{1+\frac{mcp}{hA}s}=\frac{k}{1+\tau s}$$

Q1 system consist of level tank output flow rate is directly propotional to its level drive transfer function relating out put flow rate to input flowrate and also drive transfer function relating its level to in put flowrate

M.B in unsteady state

$$q\rho_i\text{-}q_o\rho = \frac{d}{dt}(hA\rho) \;\dots\dots 1$$

M.B at steady state

$$(q_{is}\text{-}q_{os})\,\rho=0\ldots\ldots\ldots\ldots\ldots\ldots 2$$

eq1-eq2

$(q_i - q_{is})\,\rho$-$(q_o - q_{os})\,\rho$=$\frac{dq_o}{dt}\frac{A}{K}\,\rho$

q_o=kh , $Q_o = KH$

$q_o - q_{os}$=Q_o,$\frac{dq_o}{dt}$ − 0=$\frac{dQ_o}{dt}$

$q_i - q_{is}$=Q_i

$\frac{dQ_o}{dt}$=k $\frac{dH}{dt}$

$Q_i\,\rho$-$Q_o\,\rho$= $\rho\frac{dQ_o}{dt}\frac{A}{K}$

Take laplace transformation

Q_i(s) ρ - Q_o(s) ρ= $\rho\left[Q_o(s)s - Q_o(0)\right]\frac{A}{K}$

$Q_o(0)$=0

Q_i(s) ρ - Q_o(s) ρ=$Q_o(s)s\frac{A}{K}\,\rho$

Q_i(s)= $Q_o(s)s\frac{A}{K} + Q_o(s)$

Q_i(s)= Q_o(s)(1+$\frac{A}{K}s$)

G(s)=$\frac{Q_o(s)}{Q_i(s)}$=$\frac{1}{(1+\frac{A}{K}s)}$, $\frac{A}{K} = \tau, k = 1$

drive transfer function relating its level to input flow rate

M.B in unsteady state

$q\rho_i$-$q_o\rho$ =$\frac{d}{dt}$ (hAρ) … … .1

M.B at steady state

$(q_{is}-q_{os})\,\rho=0$………………2

eq1-eq2

$$(q_i - q_{is})\,\rho-(q_o - q_{os})\,\rho=\frac{dq_o}{dt}\frac{A}{K}\,\rho$$

q_o=kh , $Q_o = KH$

$$q_o - q_{os}=Q_o,\frac{dq_o}{dt} - 0 = \frac{dQ_o}{dt}$$

$$q_i - q_{is}=Q_i$$

$$\frac{dQ_o}{dt}=k\frac{dH}{dt}$$

$$Q_i\,\rho-Q_o\,\rho= \rho\,\frac{dQ_o}{dt}\frac{A}{K}$$

$$Q_i\,\rho - kH\,\rho=\frac{dH}{dt}\,\rho A$$

Take laplace transformation

$$Q_i\,(s)\rho - kH(s)\,\rho= \rho A(H(s)s + H(0))$$

$$Q_i\,(s) = H(s)[k + As]$$

$$\mathrm{G(s)}=\frac{H(s)}{Q_i\,(s)}=\frac{1}{[k+As]}=\frac{1/k}{1+s\frac{A}{k}}$$

$$\tau=\frac{A}{k},\mathrm{k}=1/k$$

Response of first order systems in series

Many physical systems can be represented by several first-order processes connected in series as shown in figure:-

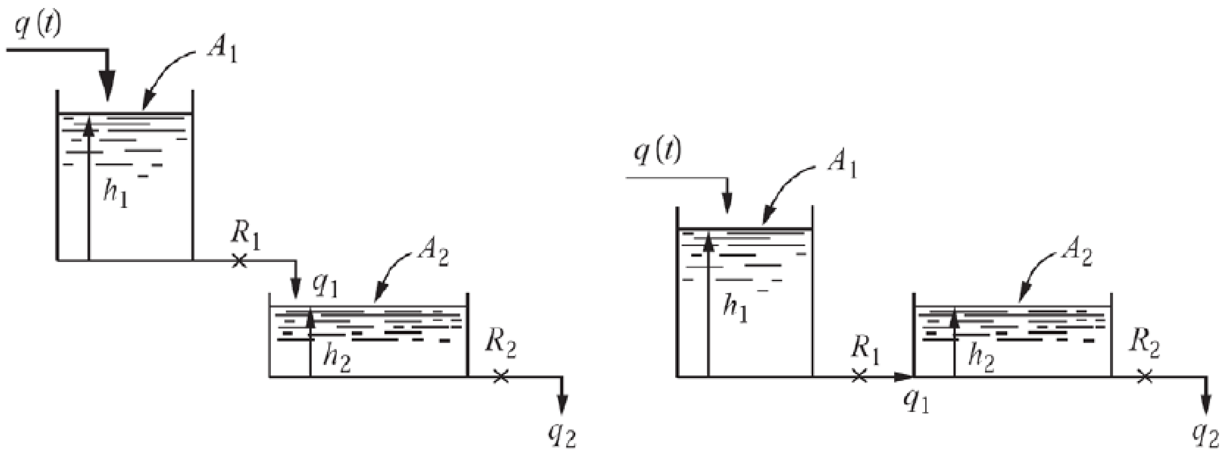
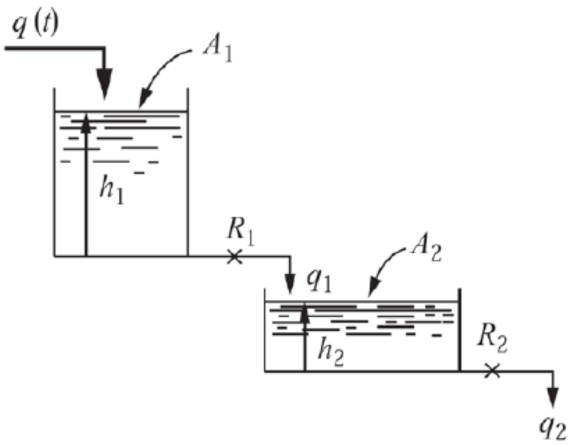


Figure 5.1 Two-tank liquid-level system: *(a)* Non-interacting; *(b)* interacting.

In fig (5.1 a) variation of h_2 does not effect on q_1 then $q_1 = \frac{h_1}{R_1}$

In fig (5.1 b) variation of h_2 does effect on q_1 then $q_1 = \frac{h_1 - h_2}{R_1}$

1-Non Interacting System



Material balance on tank 1 gives

$$q_i - \frac{h_1}{R_1} = A_1 \frac{dh_1}{dt}$$

At s.s. $q_{is} - \frac{h_{1s}}{R_1} = A_1 \frac{dh_{1s}}{dt} = 0$

By substracting both equations

$$(q_i - q_{is}) - \frac{h_1 - h_{1s}}{R_1} = A_1 \frac{d(h_1 - h_{1s})}{dt}$$

$$[Q_i - \frac{H_1}{R_1} = A_1 \frac{dH_1}{dt}] \qquad \times R_1$$

$$R_1Q_i = H_1 + R_1A_1\frac{dH_1}{dt}$$

$$\boxed{\frac{H_1(s)}{Q_i(s)} = \frac{R_1}{\tau_1 s+1}}$$ where $\tau_1 = A_1R_1$

Material balance on tank 2 gives

$$\frac{h_1}{R_1} - \frac{h_2}{R_2} = A_2\frac{dh_2}{dt}$$

At s.s. $\frac{h_{1s}}{R_1} - \frac{h_{2s}}{R_2} = A_2\frac{dh_{2s}}{dt} = 0$

By substracting both equations

$$\frac{h_1 - h_{1s}}{R_1} - \frac{h_2 - h_{2s}}{R_2} = A_2\frac{d(h_2 - h_{2s})}{dt}$$

$$\frac{H_1}{R_1} - \frac{H_2}{R_2} = A_2\frac{dH_2}{dt} \qquad \times R_2$$

$$A_2R_2\frac{dH_2}{dt} + H_2 = \frac{R_2}{R_1}H_1$$

$$\tau_2 sH_2(s) + H_2(s) = \frac{R_2}{R_1}H_1(s) \qquad \tau_2 = R_2H_2$$

$$(\tau_2 s+1)H_2(s) = \frac{R_2}{R_1}H_1(s)$$

$H_2(s) = \frac{R_2/R_1}{(\tau_2 s+1)}H_1(s)$ By substituting the lapace transform of $H_1(s)$

$$H_2(s) = \frac{R_2/R_1}{(\tau_2 s+1)} \times Q_i(s)\frac{R_1}{\tau_1 s+1}$$

$$H_2(s) = \frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}Q_i(s)$$

$$\boxed{\frac{H_2(s)}{Q_i(s)} = \frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}}$$ Non-interacting system

In the case of three non-interacting tanks in sereies the transfer function of the system will be as below:-

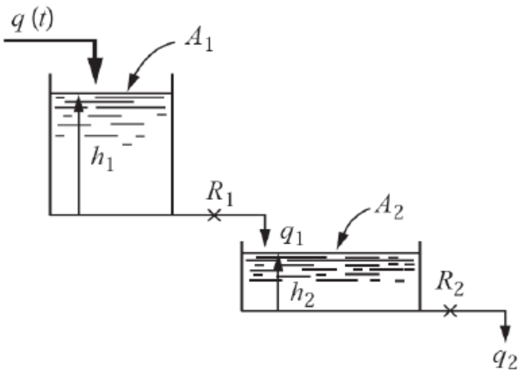
$$\frac{H_3(s)}{Q_i(s)}=\frac{R_3}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

Example:

Two non-interacting tanks are connected in series as shown in Fig. 5.1 *a*. The time constants are $\tau_2=1$ and $\tau_1=0.5$; $R_2=1$. Sketch the response of the level in tank 2 if a unit-step change is made in the inlet flow rate to tank 1.

Solution:

The transfer function for this system is found directly from Equation above thus



$$H_2(s)=\frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}Q_i(s)$$

Substituting $Q_i(s)=\frac{1}{s}$ Unit step change in Q_i

$$H_2(s)=\frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}\frac{1}{s}$$

$$=\frac{\alpha_o}{s}+\frac{\alpha_1}{(\tau_1 s+1)}+\frac{\alpha_2}{(\tau_2 s+1)}$$

$$R_2=\alpha_o(\tau_1 s+1)(\tau_2 s+1)+\alpha_1 s(\tau_2 s+1)+\alpha_2 s(\tau_1 s+1)$$

$$let\ s=0 \quad\Rightarrow\quad \alpha_o=R_2$$

$$let\ s=-\frac{1}{\tau_1}\Rightarrow\alpha_1(-\frac{1}{\tau_1})(\tau_2(-\frac{1}{\tau_1})+1)=R_2\Rightarrow\alpha_1(\frac{\tau_2}{\tau_1^2}-\frac{1}{\tau_1})=R_2\Rightarrow\alpha_1(\frac{\tau_2-\tau_1}{\tau_1^2})=R_2$$

$$\therefore\alpha_1=R_2\,(\frac{\tau_1^2}{\tau_2-\tau_1})$$

$$let\ s=-\frac{1}{\tau_2}\Rightarrow\alpha_2(-\frac{1}{\tau_2})(\tau_1(-\frac{1}{\tau_2})+1)=R_2\Rightarrow\alpha_2(\frac{\tau_1}{\tau_2^2}-\frac{1}{\tau_2})=R_2\Rightarrow\alpha_2(\frac{\tau_1-\tau_2}{\tau_2^2})=R_2$$

$$\therefore\alpha_2=R_2\,(\frac{\tau_2^2}{\tau_1-\tau_2})$$

$$H_2(s)=\frac{R_2}{s}+R_2\,(\frac{\tau_1^2}{\tau_2-\tau_1})\frac{1}{(\tau_1 s+1)}+R_2\,(\frac{\tau_2^2}{\tau_1-\tau_2})\frac{1}{(\tau_2 s+1)}$$

$$H_2(s)=R_2[\frac{1}{s}+(\frac{\tau_1\tau_2}{\tau_2-\tau_1})\frac{\tau_1}{\tau_2}\frac{1}{(\tau_1 s+1)}+(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{\tau_2}{\tau_1}\frac{1}{(\tau_2 s+1)}]$$

$$H_2(s)=R_2[\frac{1}{s}-(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{1}{\tau_2}\frac{1}{(s+1/\tau_1)}+(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{1}{\tau_1}\frac{1}{(s+1/\tau_2)}]$$

$$H_2(t)=R_2(1-(\frac{\tau_1\tau_2}{\tau_1-\tau_2})(\frac{1}{\tau_2}e^{-t/\tau_1}-\frac{1}{\tau_1}e^{-t/\tau_2}))$$

$$H_2(t) = 1 - \left(\frac{0.5}{-0.5}\right)(e^{-2t} - 2e^{-t})$$

$$\boxed{H_2(t) = 1 + e^{-2t} - 2e^{-t}}$$

$$H_1(s) = \frac{R_1}{\tau_1 s + 1} . Q_i(s)$$

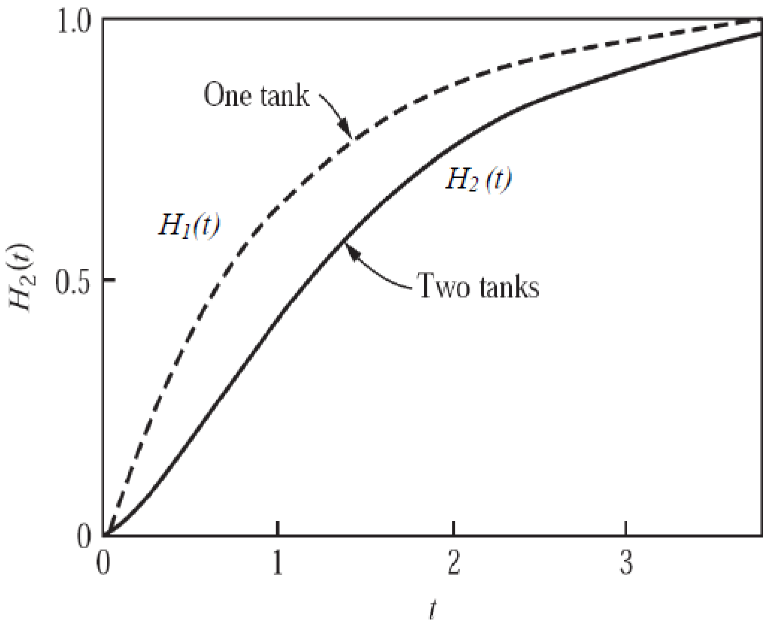
$$H_1(s) = \frac{R_1}{\tau_1 s + 1} . \frac{1}{s}$$

$$H_1(t) = R_1(1 - e^{-t/\tau_1})$$

Substitute $R_1 = 1$

$$H_1(t) = 1(1 - e^{-t/0.5})$$

$$\boxed{H_1(t) = 1 - e^{-2t}}$$



2.Interacting System

Material balance on 1st tank

$$q_i - q_1 = A_1 \frac{dh_1}{dt}$$

$$q_i - \frac{h_1 - h_2}{R_1} = A_1 \frac{dh_1}{dt}$$

Steady state

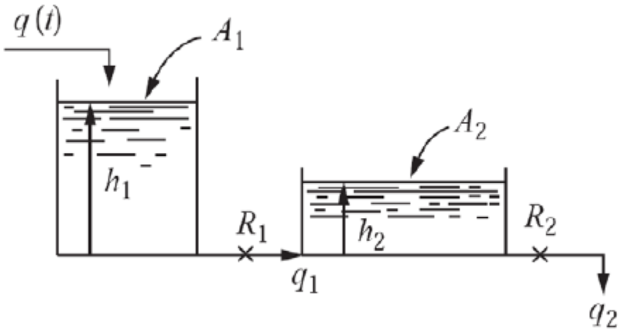
$$q_{is} - \frac{h_{1s} - h_{2s}}{R_1} = A_1 \frac{dh_{1s}}{dt} = 0$$

By substracting both equations

$$(q_i - q_{is}) - \frac{h_1 - h_{1s}}{R_1} + \frac{h_2 - h_{2s}}{R_1} = A_1 \frac{d(h_1 - h_{1s})}{dt}$$

$$[Q_i + \frac{H_2}{R_1} = \frac{H_1}{R_1} + A_1 \frac{dH_1}{dt}] \qquad \times R_1$$

$$Q_i R_1 + H_2 = H_1 + A_1 R_1 \frac{dH_1}{dt}$$



$$\tau_1 \frac{dH_1}{dt} + H_1 = Q_i R_1 + H_2$$

$$(\tau_1 s + 1) H_1(s) = R_1 Q_i(s) + H_2(s)$$

$$\boxed{H_1(s) = \frac{R_1}{(\tau_1 s + 1)} Q_i(s) + \frac{1}{(\tau_1 s + 1)} H_2(s)} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Material balance on second tank

$$\frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2} = A_2 \frac{dh_2}{dt}$$

$$\frac{h_{1s} - h_{2s}}{R_1} - \frac{h_{2s}}{R_2} = A_2 \frac{dh_{2s}}{dt} = 0$$

$$\left[\frac{H_1}{R_1} - \frac{H_2}{R_1} - \frac{H_2}{R_2} = A_2 \frac{dH_2}{dt}\right] \qquad \times R_2$$

$$A_2 R_2 \frac{dH_2}{dt} + H_2 = \frac{R_2}{R_1}(H_1 - H_2)$$

$$(\tau_2 s + 1) H_2(s) = \frac{R_2}{R_1}(H_1(s) - H_2(s)) \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Substituting for $H_1(s)$ from eq(1) in eq(2)

$$(\tau_2 s + 1) H_2(s) = \frac{R_2}{R_1}\left[\frac{R_1}{(\tau_1 s + 1)} Q_i(s) + \frac{1}{(\tau_1 s + 1)} H_2(s) - H_2(s)\right]$$

$$\left[(\tau_2 s + 1) H_2(s) = \frac{R_2 Q_i(s)}{(\tau_1 s + 1)} + \frac{R_2}{R_1}\frac{H_2(s)}{(\tau_1 s + 1)} - \frac{R_2}{R_1} H_2(s)\right] \qquad \times (\tau_1 s + 1)$$

$$(\tau_2 s+1)(\tau_1 s+1)H_2(s)=R_2Q_i(s)+\frac{R_2}{R_1}H_2(s)-\frac{(\tau_1 s+1)R_2}{R_1}H_2(s)]$$

$$(\tau_1\tau_2 s^2+\tau_1 s+\tau_2 s+1)H_2(s)+\frac{\tau_1 R_2 s}{R_1}H_2(s)=R_2Q_i(s)$$

Let $\frac{\tau_1 R_2}{R_1}=\frac{A_1R_1R_2}{R_1}=A_1R_2=\tau_{12}$

$$(\tau_1\tau_2 s^2+(\tau_1+\tau_2+\tau_{12})s+1)H_2(s)=R_2Q_i(s)$$

$$\boxed{H_2(s)=\frac{R_2}{\tau_1\tau_2 s^2+(\tau_1+\tau_2+\tau_{12})s+1}.Q_i(s)}$$ Interacting system

$$\boxed{H_2(s)=\frac{R_2}{\tau_1\tau_2 s^2+(\tau_1+\tau_2)s+1}.Q_i(s)}$$ Non- Interacting system

The difference between the transfer function for the non-interacting system, and that for the interacting system, is the presence of the cross-product term A_1R_2 in the coefficient of s. $\tau 1=A1R1$

Example:

To understand the effect of interaction on the transient response of a system, consider a two-tank system for which the time constants are equal ($\tau_1=\tau_2=\tau$).

$\tau_1=\tau_2=\tau_{12}=\tau$

$Q_2(t)$=? Output flow rate

$$Q_i(s)=\frac{1}{s}$$

Solution:

Non-interacting system

$$\frac{H_2(s)}{Q_i(s)}=\frac{R_2}{\tau_1\tau_2 s^2+(\tau_1+\tau_2)s+1} \qquad \tau_1=\tau_2$$

$$\frac{H_2(s)}{Q_i(s)}=\frac{R_2}{\tau^2 s^2+2\tau s+1} \quad \text{but } Q_2(s)=\frac{H_2(s)}{R_2}$$

$$\frac{Q_2(s)}{Q_i(s)}=\frac{1}{\tau^2 s^2+2\tau s+1}=\frac{1}{(\tau s+1)(\tau s+1)}=(\frac{1}{\tau s+1})^2$$

If $Q_i(s)=\frac{1}{s}$

$$Q_2(s)=\frac{1}{(\tau s+1)^2}\cdot\frac{1}{s}=\frac{\alpha_o}{s}+\frac{\alpha_1}{(\tau s+1)^2}+\frac{\alpha_2}{\tau s+1}$$

By multiplying both sides by $s(\tau s+1)^2$ and expanding, we get

$$\alpha_o(\tau s+1)^2+\alpha_1 s+\alpha_2 s(\tau s+1)=1$$

$$\alpha_o(\tau^2 s^2+2\tau s+1)+\alpha_1 s+\alpha_2(\tau s^2+s)=1$$

$$s^2(\alpha_o\tau^2+\alpha_2\tau)+s(2\tau\alpha_o+\alpha_1+\alpha_2)+\alpha_o=1$$

$$s^0 \quad \Rightarrow \alpha_o=1$$

$$s^2 \quad \Rightarrow \alpha_o\tau^2+\alpha_2\tau=0 \Rightarrow \tau^2+\alpha_2\tau=0 \Rightarrow \alpha_2=-\tau$$

$$s^1 \quad \Rightarrow 2\alpha_o\tau+\alpha_1+\alpha_2=0 \Rightarrow 2\tau+\alpha_1-\tau=0 \Rightarrow \alpha_1=-\tau$$

$$Q_2(s)=\frac{1}{s}-\frac{\tau}{(\tau s+1)^2}-\frac{\tau}{\tau s+1}$$

$$Q_2(s)=\frac{1}{s}-\frac{\tau}{(\tau s+1)^2}-\frac{\tau}{\tau s+1}$$

$$Q_2(s)=\frac{1}{s}-\frac{1}{\tau}\frac{1}{(s+1/\tau)^2}-\frac{1}{s+1/\tau}$$

$$\boxed{Q_2(t)=1-e^{-t/\tau}-\frac{t}{\tau}e^{-t/\tau}}$$ for non-interacting

Interacting system

If the tanks are interacting, the overall transfer function, according to Equation of interacting system (assuming further that $A_1=A_2$)

$$Q_2(s)=\frac{1}{\tau^2 s^2+3\tau s+1}\cdot\frac{1}{s}$$

By application of the quadratic formula, the denominator of this transfer function can be written as

$$Q_2(s)=\frac{1}{s}\frac{1}{(0.38\tau s+1)(2.62\tau s+1)}$$

$$Q_2(s)=\frac{\alpha_o}{s}+\frac{\alpha_1}{0.38\tau s+1}+\frac{\alpha_2}{2.62\tau s+1}$$

$$let\ s=0 \Rightarrow \alpha_o=1$$

$$let\ s=-\frac{1}{0.38\tau} \Rightarrow \alpha_1=-0.38\tau\frac{1}{2.62\tau(-\frac{1}{0.38\tau})+1}=0.0664\tau$$

$$let\ s=-\frac{1}{2.62\tau} \Rightarrow \alpha_1=-2.62\tau\frac{1}{0.38\tau(-\frac{1}{2.62\tau})+1}=-3.664\tau$$

$$Q_2(s)=\frac{1}{s}+\frac{0.0664\tau}{0.38\tau s+1}-\frac{3.0664\tau}{2.62\tau s+1}$$

$$Q_2(s)=\frac{1}{s}+\frac{0.0664\tau/0.38\tau}{s+1/0.38\tau}-\frac{3.0664\tau/2.62\tau}{s+1/2.62\tau}$$

$$Q_2(s)=\frac{1}{s}+\frac{0.17}{s+1/0.38\tau}-\frac{1.17}{s+1/2.62\tau}$$

$$\boxed{Q_2(t)=1+0.17e^{-t/0.38\tau}-1.17e^{-t/2.26\tau}}$$

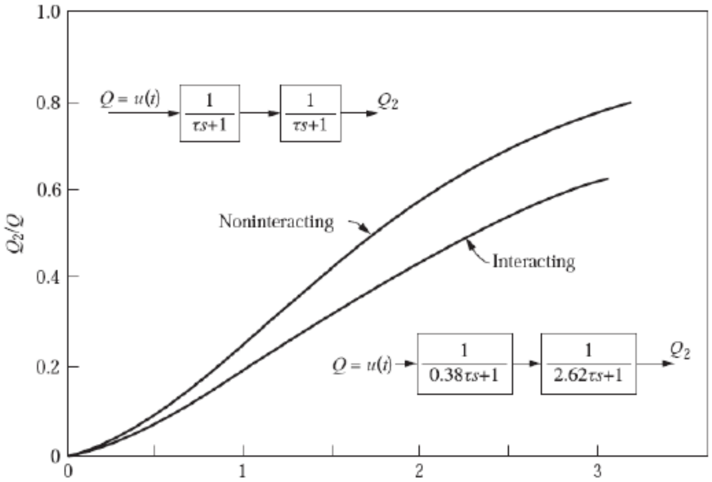


Figure: Effect of interaction on step response of two tank system.

Q System consist of two identical non interacting level tank that connected in series output flow rate from each tank is directly proportional to square root of its level drive system transfer function relating output flow rate to input flow rate

M.B in unsteady state

$q\rho_i$-$q_o\rho$ =$\frac{d}{dt}$(hAρ)

$q_o = k\sqrt{h}$,q_o^2=hR

$q_i\rho$-$q_o\rho$=$\frac{dq_o^2}{dt}R^2$A ρ

q_i-q_o=2R^2A $\rho\frac{dq_o}{dt}$

$\frac{q_i}{q_o}$-1=2R^2A $\frac{dq_o}{dt}$……1

Linerazation $\frac{q_i}{q_o}$

$Z = z_s + dz_s$

$\frac{q_{is}}{q_{os}} + (\frac{1}{q_{os}} Q_i - \frac{q_{is}}{q_{os}^2} Q_o)$ sub in 1

$\frac{q_{is}}{q_{os}} + (\frac{1}{q_{os}} Q_i - \frac{q_{is}}{q_{os}^2} Q_o) - 1 = 2R^2 A\, \rho \frac{dq_o}{dt}$……..2

$q_o - q_{os} = Q_o$

$dq_o - 0 = dQ_o$

M.B at steady state

$q_{is} - q_{os} = 0$

$\frac{q_{is}}{q_{os}} - 1 = 0$….3

Eq2-eq3

$\frac{1}{q_{os}} Q_i - \frac{q_{is}}{q_{os}^2} Q_o = 2R^2 A\, \rho \frac{dQ_o}{dt}$

Take laplace transformation

$\frac{1}{q_{os}} Q_i(s) - \frac{q_{is}}{q_{os}^2} Q_o(s) = 2R^2 A\, \rho Q_o(s) s$

$\frac{1}{q_{os}} Q_i(s) = \frac{q_{is}}{q_{os}^2} Q_o(s) + 2R^2 A\, \rho Q_o(s) s$

$\frac{1}{q_{os}} Q_i(s) = Q_o(s)(\frac{q_{is}}{q_{os}^2} + 2R^2 A\, s)$

$G1(s) = \frac{Q_o(s)}{Q_i(s)} = \frac{1/q_o}{\frac{q_{is}}{q_{os}^2} + 2R^2 A\, s} = \frac{\frac{q_{os}}{q_{is}}}{1 + \frac{2AR^2 q_{os}^2}{q_{is}} s}$ for the first tank

$K = \frac{q_{os}}{q_{is}}, \tau = \frac{2AR^2 q_{os}^2}{q_{is}}$ similarity for the second tank

$$G2(s)=\frac{\frac{q_{1s}}{q_{os}}}{1+\frac{2AR^2q_{1s}^2}{q_{os}}s}$$

G(s)= G1(s). G2(s)

$$, G(s)=\frac{\frac{q_{os}}{q_{is}}}{1+\frac{2AR^2q_{os}^2}{q_{is}}s}\cdot\frac{\frac{q_{1s}}{q_{os}}}{1+\frac{2AR^2q_{1s}^2}{q_{os}}s}$$

Q level tank system of 4 m^2 area in which the out let flow rate is proportional to level square root is at steady state when q_o=q_i=8m^3/minand h=4 drive system transfer function relating its level to input flow rate

M.B in unsteady state

$$q_i\rho\text{-}q_o\rho=\frac{d}{dt}(hA\rho)\;\dots\dots1$$

$$q_o=k\sqrt{h},h\text{-}h_s=H,\ \frac{dh}{dt}=k\,\frac{dH}{dt}$$

$$q\rho_i\text{-}k\sqrt{h}\rho=\frac{d}{dt}(hA\rho)$$

Linearization z

$$Z=z_s+dz_s$$

$$z_s=kh_s^{1/2},\ dz_s=\frac{1}{2}kh_s^{-1/2}H$$

$$q_i-(\ kh_s^{1/2}+\frac{1}{2}kh_s^{-1/2}H)=\frac{dh}{dt}(A\rho)\dots..2$$

M.B at steady state

$$q_{is}-q_{os}=0$$

$$q_{is}-kh_s^{1/2}=0\dots..3$$

Eq 2-eq 3

$(q_i - q_{is}) - \frac{1}{2} k h_s^{-1/2} H = \frac{dH}{dt} A$ take Laplace transformation

$$Q_i(s) - \frac{1}{2} k h_s^{-1/2} H(s) = A(H(s)s + H(0))$$

$$Q_i(s) = H(s)(As + \frac{1}{2} k h_s^{-1/2})$$

$$G(s) = \frac{H(s)}{Q_i(s)} = \frac{1}{As + \frac{1}{2} k h_s^{-1/2}} = \frac{1/A}{s + \frac{1}{2A} k h_s^{-1/2} s}$$

$K = 1/A, \tau = \frac{1}{2A} k h_s^{-1/2}$, $q_o = k\sqrt{h}, k = \frac{q_o}{\sqrt{h}}, k = \frac{8}{2} = 4$

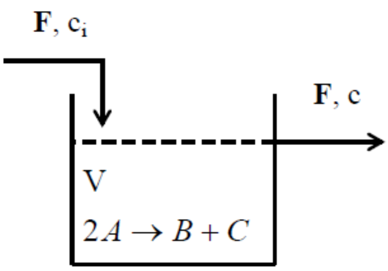
$$\tau = \frac{1}{2*4} 4 * 8^{-1/2} = 1/4\sqrt{2}$$

Example: Mixing tank with chemical reaction

$2A \rightarrow B + C$

$Reaction\ Rate = r = -kc^2$

$G(s) = \frac{C(s)}{C_i(s)} = ?$



c_i, c : Composition of component (A)

V: Constant=L

F: Constant=L/min

Solution:

In - out - rate of reaction = accumulation

$Fc_i - Fc - Vkc^2 = V\frac{dc}{dt}$ Un-steady state

$c^2 = c_s^2 + 2c_s(c - c_s)$

$Fc_i - Fc - Vk[c_s^2 + 2c_s(c - c_s)] = V\frac{dc}{dt}$

$Fc_{is} - Fc_s - Vkc_s^2 = V\frac{dc_s}{dt} = 0$ Steady state $c = c_s$

$$F(c_i - c_{is}) - F(c - c_s) - Vk[2c_s(c - c_s)] = V\frac{d(c - c_s)}{dt}$$

$$V\frac{dC}{dt} + (F + 2Vkc_s)C = FC_i \qquad \div (F + 2Vkc_s)$$

$$\tau\frac{dC}{dt} + C = RC_i$$

$$\boxed{\frac{C(s)}{C_i(s)} = \frac{R}{\tau s + 1}} \qquad \text{1st order system}$$

Where $C = c - c_s \quad , \quad C_i = c_i - c_{is}$

$$\tau = \frac{V}{F + 2Vkc_s} \quad , \quad R = \frac{F}{F + 2Vkc_s}$$

where $\quad Y = c - c_s$
$\quad X = F - F_s$

$$[V\frac{dY}{dt} + F_s Y = (c_i - c_s)X] \qquad \div F_s$$

$$\frac{V}{F_s}\frac{dY}{dt} + Y = \frac{(c_i - c_s)}{F_s}X$$

$$\tau\frac{dY}{dt} + Y = RX$$

Where

$$\tau = \frac{V}{F_s} \quad , \quad R = \frac{(c_i - c_s)}{F_s}$$

$$\boxed{\frac{Y(s)}{X(s)} = \frac{R}{\tau s + 1}} \qquad 1^{st} \text{ order system}$$

Q liquid flows continuously through a constant volume vessel so that input and output flow rate are equal an electric heater supplies heating rate to the liquid a-drive transfer function relating outlet temp to inlet temp

b- drive transfer function relating out let temp to mass flow rate a-unsteady state M.B

mcp$(t_i - t_r)$-mcp$(t_o - t_r)$+q=$\frac{d}{dt}(v\rho \text{cp}(t_o - t_r))$….1

steady state balance

mcp$(t_{is} - t_r)$-mcp$(t_{os} - t_r)$+q=0……..2

eq 1-eq 2

mcp$(t_i - t_{is})$-mcp$(t_o - t_{os})$+q=$\frac{dt_o}{dt} v\rho$cp

$t_i - t_{is} = T_i, t_o - t_{os}$=$T_o$, $\frac{dt_o}{dt}$=$\frac{dT_o}{dt}$ sub in eq

mcpT_i-mcpT_o+q=$\frac{dT_o}{dt} v\rho$cp take Laplace transformation

mcp$T_i(S)$-mcp$T_o(S)$=$(T_o(S)s + T_o(0))v\rho$cp

mcp$T_i(S)$=mcp$T_o(S) + T_o(S)sv\rho$cp

mcp$T_i(S) = T_o(S)$(mcp+vρcp)

G(s)=$\frac{T_o(S)}{T_i(S)}$=$\frac{m}{m+v\rho}$=$\frac{1}{\frac{v}{m}\rho s+1}$

B-un steady state balance

mcp$(t_i - t_r)$-mcp$(t_o - t_r)$+q=$\frac{d}{dt}(v\rho \text{cp}(t_o - t_r))$….1

mcpt_i-mcpt_o+q=$\frac{dt_o}{dt} v\rho$cp….1

linerazation

z=mt_o,Z_s=$m_s t_{os}$,dZ_s=$m_s(t_o - t_{os})$+ t_{os}(m-m_s) sub in eq

mcpt_i-$(m_s t_{os} cp + m_s(t_o - t_{os})cp$+ t_{os}(m-m_s)cp) +q=$\frac{dt_o}{dt} v\rho$cp….2

steady state M.B

$m_s$cpt_{is}-$m_s$cpt_{os}+q=0…..3

Eq2 –eq 3

(m-m_s) t_{is}-$m_s(t_o - t_{os})$- t_{os}(m-m_s)=v$\rho \frac{dt_o}{dt}$

M= m-m_s,$T_o = t_o - t_{os}$sub in eq

Mt_{is}-$m_s T_o$-t_{os}M= v$\rho \frac{dT_o}{dt}$

Take laplace transformation

M$(s)t_{is}$-$m_s T_o(s)$-t_{os}M(s)= v$\rho(T_o(s)$s-$T_o(0))$

M$(s)(t_{is}$-$t_{os})$= $T_o(s)$(vρs+1)

G(s)=$\frac{T_o(s)}{M(s)}=\frac{(t_{is}-t_{os})}{v\rho s+m_s}=\frac{\frac{(t_{is}-t_{os})}{m_s}}{\frac{v\rho}{m_s}s+1}$

Transfer function for multiple input

Q liquid flows continuously through a constant volume vessel so that input and output flow rate are equal an electric heater supplies heating rate to the liquid a-drive transfer function relating outlet temp to inlet temp and heating rate

unsteady state M.B

mcp$(t_i - t_r)$-mcp$(t_o - t_r)$+q=$\frac{d}{dt}(v\rho$cp$(t_o - t_r))$….1

steady state balance

mcp$(t_{is} - t_r)$-mcp$(t_{os} - t_r)$+q_s=0……..2

eq 1 –eq 2

mcp$(t_i - t_{is})$-mcp$(t_o - t_{os})$+(q-q_s)=$\frac{dt_o}{dt} v\rho$cp

mcpT_i-mcp T_o+Q=$\frac{dT_o}{dt} v\rho$cp

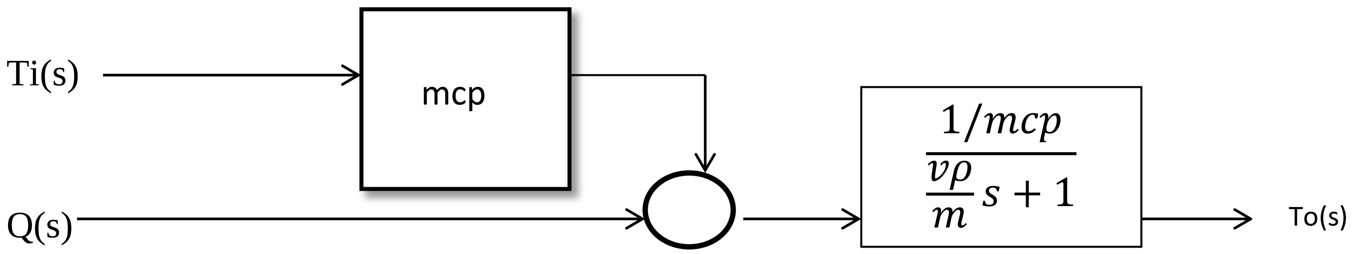
mcp$T_i(s)$-mcp $T_o(s)$+Q(s)= $v\rho$cp$\left(T_o(s)s + T_o(0)\right)$

$T_o(s)$(mcp+ $v\rho$cps)= $T_i(s)$mcp+ Q(s)

Y(s)= $T_o(s)$,X(s)= $T_i(s)$mcp+ Q(s)

T_o (s)=$\frac{1}{1+\frac{v\rho}{m}s}$($T_i(s)$mcp+ Q(s),G(s)= $\frac{1}{1+\frac{v\rho}{m}s}$ if the q is cooling

mcp$(t_i - t_r)$-mcp$(t_o - t_r)$-q=$\frac{d}{dt}(v\rho$cp$(t_o - t_r))$…



for more complex case drive transfer function relating out put temp to input temp ,mass flow rate and heating rate

un steady state balance

mcp$(t_i - t_r)$-mcp$(t_o - t_r)$+q=$\frac{d}{dt}(v\rho$cp$(t_o - t_r))$….

linerazation

z=mt_o,Z_s=$m_s t_{os}$,dZ_s=$m_s(t_o - t_{os})$+ t_{os}(m-m_s)

z1=mt_i,$Z1_s$=$m_s t_{is}$,d$Z1_s$=$m_s(t_i - t_{is})$+ t_{is}(m-m_s)

sub in eq

$m_s t_{is}$cp $+ m_s cp(t_i - t_{is}) +\ t_{is}$(m $- m_s$)cp) $- m_s t_{os} cp + m_s(t_o - t_{os})cp$+ t_{os}(m-m_s)cp+q=$\frac{d(t_o}{dt}(v\rho$cp….1

Steady sate balance

$m_s$cpt_{is} −-$m_s$cpt_{os}+q_s =0…..2

Eq 1-eq2

$m_s cp(t_i - t_{is}) +\ t_{is}$(m $- m_s$)cp) $-$ ($m_s(t_o - t_{os})cp$+ t_{os}(m-m_s)cp)+ q=$\frac{d(t_o}{dt}(v\rho$cp

$m_s cp T_i +\ t_{is} M$-$m_s T_o cp - t_{os}$Mcp+(q-q_s)= $v\rho$cp$\frac{dT_o}{dt}$

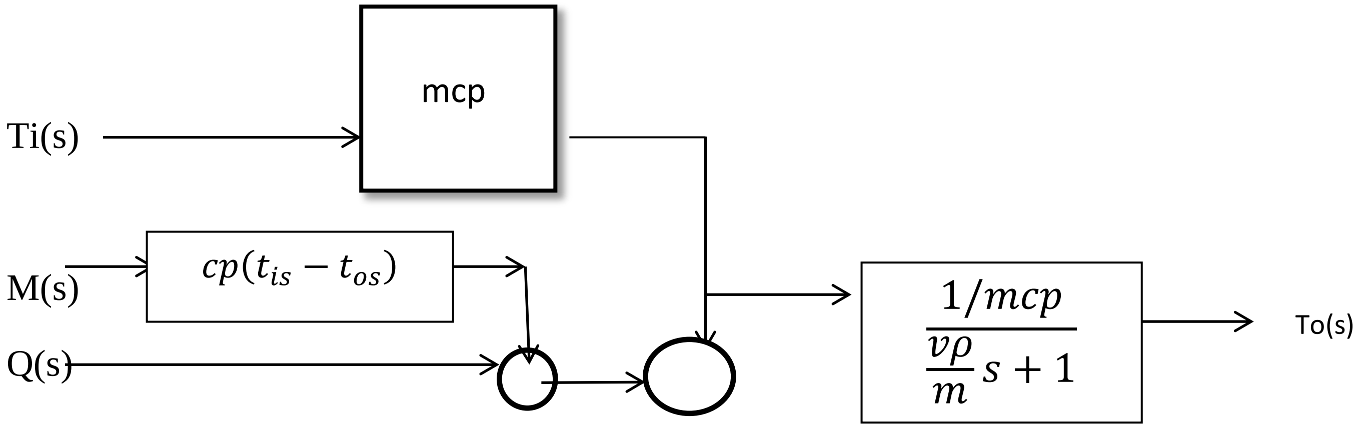
Take laplace transformation

$m_s cpT_i(s) + \ t_{is}M(s)cp\text{-}m_sT_o(s)cp - t_{os}\text{Mcp+Q=}\ v\rho \text{cp}T_o(s)s$

$m_s cpT_i(s) + M(s)cp(t_{is} - t_{os}) + Q(s) = T_o(s)(v\rho \text{cps}+m_s cp)$

$$T_o(s) = \frac{1}{(v\rho \text{cps} + m_s cp)} m_s cpT_i(s) + M(s)cp(t_{is} - t_{os}) + Q(s)$$

$$T_o(s) = \frac{1/m_s}{(\frac{v\rho}{m_s}\text{s} + 1)} m_s cpT_i(s) + M(s)cp(t_{is} - t_{os}) + Q(s)$$



Home Work

Q 1system consist of two identical non interacting level that connected in series out put flow rate from each tank is its level by the equation q=18 $h^{2/3}$ drive system transfer function relating its level to input flow rate.Liquid level in each tank 8 m the cross sectional of each tank 12 m^2,15 m^2respectively

Q2 liquid at temperature ti flows in stirred tank cooling tank where its temp where drops to t1by means of cooling water in coil with transfer rate q=UA$(t_1 - t_w)$ the out let flows in the second tank where its temp drops to t2 by means of an evaporating refrigeration a coil with transfer rate q=$m_R\lambda_R$

Q3 water flows into constant –level stirred tank of 50 m3 volume and exit at the same volumetric flow rate of 2 m3/min at steady state the tank is fitted with an electric cooler that operate at constant rate to cool the water from 40 c to 10 c at steady state drive transfer function relating out let temp to volumetric flow rate

Q4 the second order reaction 2A $\longrightarrow$ B is carried out in CSTR find the transfer function relating output concentration to input concentration

Q5 Two identical constant level stirred tank reactors are connected in series the reaction taking place in each reactor is second order drive the system transfer function relating output reactant concentration from second tank to A-input reactant concentration to first reactor

B-both of input reactant concentration and flow rate to first tank

Q6 in continuous biochemical reactor the volumetric reaction rate r=$\frac{\alpha C}{B+C}$ drive the transfer function relating out conc to inlet conc.

Response of first order system

1- Step

Y(s)=X(s).G(s)

Y(s)=$\frac{A}{S}.\frac{k}{\tau s+1}=\frac{\frac{Ak}{\tau}}{s(s+\frac{1}{\tau})}=\frac{a}{s}+\frac{b}{(s+\frac{1}{\tau})}$,a=Ak

=a(S+$\frac{1}{\tau}$)+bs

a=Ak,Ak+b=0,b=-Ak

s a+b

s^0 $\frac{a}{\tau}$

Y(s)=$\frac{Ak}{S}-\frac{Ak}{s+\frac{1}{\tau}}$

Y(s)=Ak($\frac{1}{s}$-$\frac{1}{s+\frac{1}{\tau}}$)

Y(t)=Ak(1-$e^{-\frac{t}{\tau}}$)

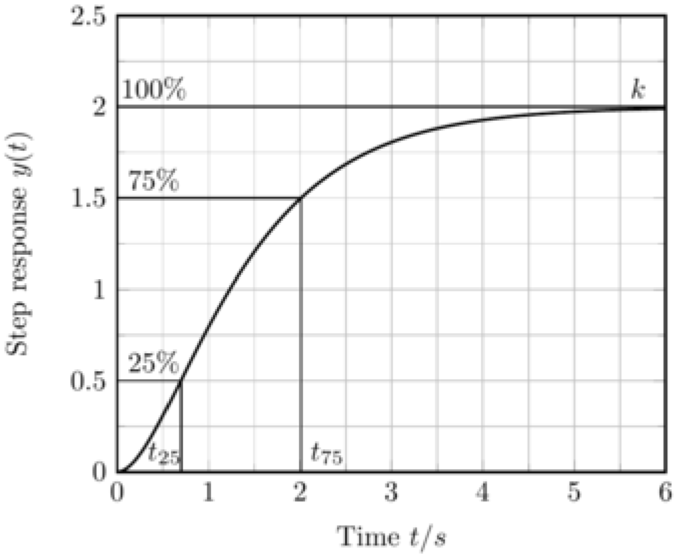
To find extreme

Lim(0)=$\lim_{t=0} y(t) = \lim_{s=\infty} s\,\frac{Ak}{s(\tau s+1)} = 0$

$$Y(\infty)=\lim_{t=\infty} y(t)=\lim_{s=0} s\,\frac{Ak}{s(\tau s+1)}=Ak$$

For $Y(t)=Ak(1-e^{-\frac{t}{\tau}})$

$Y(0)=0$ and $Y(\infty)=Ak$



Example:

A thermometer having a time constant of 0.1 min is at a steady state temperature of 90 F°. At time t = 0, the thermometer is placed in a temperature bath maintained at 100°F. Determine the time needed for the thermometer to read 98 F°.

Solution:

At s.s. $x_s=y_s=90$ F°

Step change $X(s)=\frac{A}{s}$

A=100-90=10

$$X(s)=\frac{10}{s}$$

$$Y(s)=\frac{1}{\tau s+1}\frac{A}{s}=\frac{1}{0.1s+1}\frac{10}{s}=\frac{10}{s(0.1s+1)}=\frac{10}{0.1s(s+10)}=\frac{A}{0.1s}+\frac{B}{s+10}$$

$$A(s+10)+B(0.1s)=10$$

$$s=0\Rightarrow A=\frac{10}{10}=1$$

$$s=-10\Rightarrow B=-10$$

$$Y(s)=\frac{1}{0.1s}-\frac{10}{s+10}=\frac{10}{s}-\frac{10}{s+10}$$

By taken laplace inverse for the equation

$Y(t)=10-10e^{-10t}=10(1-e^{-10t})$

Substitute $Y(t)=y(t)-y_s=98-90$

$Y(t)=8$

$8=10(1-e^{-10t})$

$0.8=1-e^{-10t}$

$ln(e^{-10t})=ln(0.2)$

$-10t=ln(0.2)$

$t=-\ln(0.2)\times 0.1$

t=0.161 min

2-Ramp Response

Y(s)=X(s).G(s)

$$=\frac{A}{s^2}\cdot\frac{k}{\tau s+1}=\frac{\frac{Ak}{\tau}}{s^2(s+\frac{1}{\tau})}=\frac{a}{s^2}+\frac{b}{s}+\frac{c}{s+\frac{1}{\tau}}=a(s+\frac{1}{\tau})+bs\left(s+\frac{1}{\tau}\right)+cs^2$$

$$=as+a\frac{1}{\tau}+bs^2+bs\frac{1}{\tau}+cs^2$$

$s^2 \qquad (b+c)=0$

$S \qquad a+\frac{b}{\tau}=0$

$s^0 \qquad a\frac{1}{\tau}=\frac{Ak}{\tau}= a=Ak$

$Ak+\frac{b}{\tau} \quad =0$

$B=-Ak\tau$

$-Ak\tau+c=0$

$C=Ak\tau$

$$=\frac{Ak}{s^2}-\frac{Ak\tau}{s}+\frac{Ak\tau}{S+\frac{1}{\tau}}$$

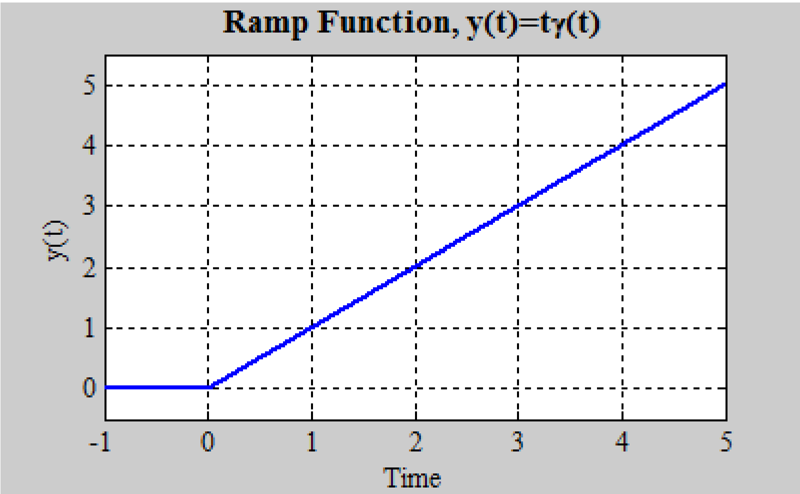
$$Y(0)=\lim_{t=0} y(t) = \lim_{s=\infty} s \frac{Ak}{s^2(s\tau+1)}=0$$

$$Y(\infty)=\lim_{t=\infty} y(t) = \lim_{s=0} s \frac{Ak}{s^2(s\tau+1)} =\infty$$

$$Y(s)= \frac{Ak}{s^2} - \frac{Ak\tau}{s} + \frac{Ak\tau}{S+\frac{1}{\tau}}$$

$$Y(t)=Ak\left[t - \tau + \tau e^{-\frac{t}{\tau}}\right]$$

$Y(0)=0, Y(\infty)=\infty$



3-impluse

Y(s)= X(s).G(s)

$$Y(S)=A\frac{k}{\tau s+1}=\frac{\frac{Ak}{\tau}}{S+\frac{1}{\tau}}=\frac{Ak}{\tau} e^{-\frac{t}{\tau}}$$

To find extrems

$$Y(0)=\lim_{t=0} Y(t) = \lim_{s=\infty} s \frac{k}{\tau s+1}=s\frac{\frac{Ak}{\tau}}{S+\frac{1}{\tau}} =\frac{Ak}{\tau}$$

$$Y(0)= \frac{Ak}{\tau}, Y(\infty)=0$$

4-sinsoidal response

$Y(s)=\frac{A}{s^2+w^2}w.\frac{k}{\tau s+1}$

$=\frac{as+b}{s^2+w^2}+\frac{c}{\tau s+1}=(as+b)(\tau s+1)+c(s^2+w^2)=$

$=a\tau s^2+as+b\tau s+b+cs^2+cw^2$

$s^2 \quad a\tau+c=0$

$S \quad a+b\tau=0$

$s^0 \quad b+cw^2=Akw$

$B=Akw-cw^2=a+(Akw-cw^2)\tau=0$

$a+Akw\tau-c\tau w^2=0$

$a=c\tau w^2-Akw\tau$

$(c\tau w^2-Akw\tau)\tau+c=0$

$(cw^2\tau^2-Akw\tau^2)+c=0$

$C(1+w^2\tau^2)=Ak\tau^2 w$

$C=\frac{Ak\tau^2 w}{1+w^2\tau^2}$

$a=\frac{Ak\tau^2 w}{1+w^2\tau^2}\tau w^2-Akw\tau$

$a=\frac{Ak\tau^2 w}{1+w^2\tau^2}\tau w^2-\frac{Akw\tau}{1+w^2\tau^2}1+w^2\tau^2$

$a=\frac{Ak\tau^2 w}{1+w^2\tau^2}\tau w^2-\frac{Akw\tau+Ak}{1+w^2\tau^2}w\tau w^2\tau^2$

$a=\frac{-Akw\tau}{1+w^2\tau^2}, b=Akw-cw^2$

$a+b\tau=0, a=-b\tau, b=\frac{-a}{\tau}, b=\frac{Akw}{1+w^2\tau^2}$

$=\frac{\left(\frac{-Akw\tau}{1+w^2\tau^2}\right)s+\frac{Akw}{1+w^2\tau^2}}{s^2+w^2}+\frac{\frac{Ak\tau^2 w}{1+w^2\tau^2}}{\tau s+1}$

$Y(s)=\frac{Ak}{1+w^2\tau^2}\left[\frac{-w\tau}{s^2+w^2}s+\frac{w}{s^2+w^2}+\frac{w\tau^2}{\tau s+1}\right]$

$$Y(s)=\frac{Ak}{1+w^2\tau^2}\left[-w\tau coswt+sinwt+\tau^2we^{-\frac{t}{\tau}}\right]$$

$$r^2=p^2+q^2, r^2=\sqrt{\left(\frac{Ak}{1+w^2\tau^2}\right)^2(-w\tau)^2+\left(\frac{Ak}{1+w^2\tau^2}\right)^2(1)^2}$$

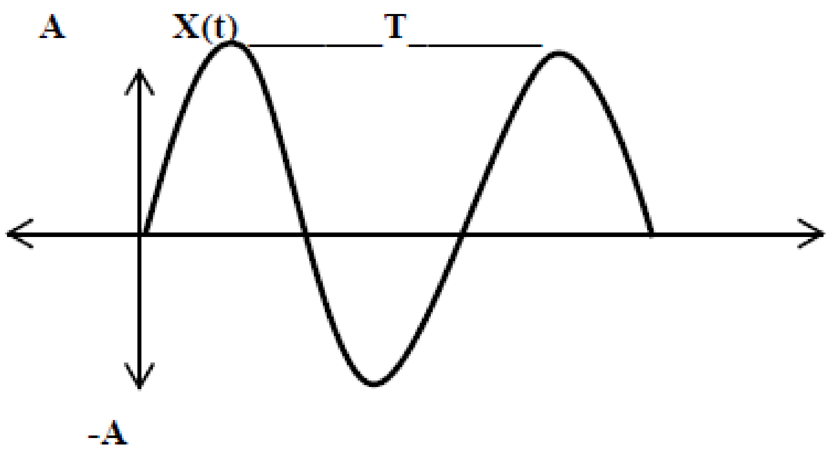
$$r=\frac{Ak}{1+w^2\tau^2}\sqrt{1+w^2\tau^2}=\frac{Ak}{\sqrt{1+w^2\tau^2}}, \Theta=\tan^{-1}(-\tau w)$$

$$Y(t)=\frac{Ak}{\sqrt{1+w^2\tau^2}}\sin[wt+\tan^{-1}(-\tau w)]$$

X=Asinwt

Amplitude ratio

$$AR=\frac{\frac{Ak}{\sqrt{1+w^2\tau^2}}}{A}=\frac{k}{\sqrt{1+w^2\tau^2}}$$



By comparing Eq. for the input forcing Y(t) function with Eq. for the ultimate periodic response X(t), we see that 1. The output is a sine wave with a frequency w equal to that of the input signal. 2. The ratio of output amplitude to input amplitude is 111$_{22}$<τ+*w*. 3. The output lags behind the input by an angle φ. It is clear that lag occurs, for the sign of φ is always negative.

Θ>1 phase lag

Θ<1 phase load

Example:

A mercury thermometer having a time constant of 0.1 min is placed in a temperature bath at 100°F and allowed to come to equilibrium with the bath. At time t= 0, the temperature of the bath begins to vary sinusoidally about its average temperature of 100°F with an amplitude of 2°F If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function of time. What is the phase lag?

In terms of the symbols used in this chapter

$\tau = 0.1$

$t < 0 \qquad x_s = y_s = 100$

$t \geq 0 \qquad x(t) = 100 + 2\,sin(wt)$

$f = \dfrac{10}{\pi}$

Solution

$$w = 2\pi f = 2\pi \times \frac{10}{\pi} = 10 \;\; rad / \text{min}$$

$$T = \frac{1}{f} = \frac{10}{\pi} \;\; \text{min/} \, cycle$$

$$X(t) = x(t) - x_s = 100 + 2\,sin\,20t - 100$$

$$X(t) = 2\,sin\,20t$$

$$X(s) = \frac{2 \times 20}{s^2 + 20^2}$$

Ultimate response $t \to \infty \qquad then \qquad e^{-t/\tau} = 0$

$$Y(t) = \frac{A}{\sqrt{1 + w^2\tau^2}} sin(wt + \phi)]$$

$$\phi = tan^{-1}(-w\tau) = tan^{-1}(-20 \times 0.1) = tan^{-1}(-2)$$

$$\phi = -63.5^o$$

Ultimate response at the above angle

$$Y(t) = \frac{2}{\sqrt{1 + (0.1 \times 20)^2}} sin(20t - 63.5)]$$

$$Y(t)=\frac{2}{\sqrt{5}}sin(20t-63.5)]$$

$$Y(t)=0.896sin(20t-63.5)]$$

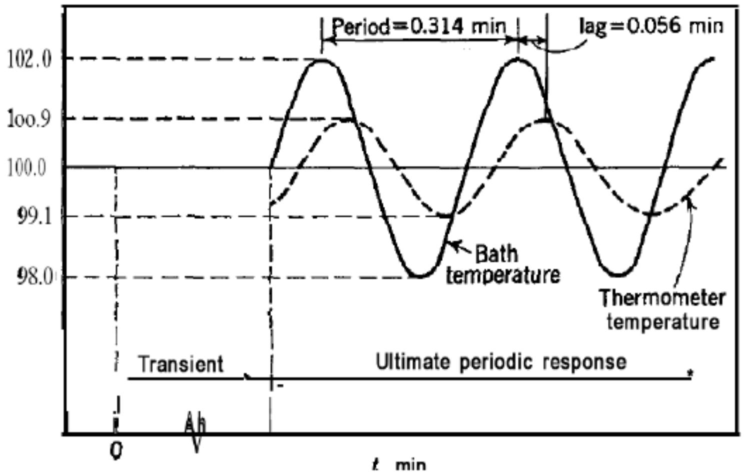
Ultimate response

In general, the lag in units of time is given by:-

$$phase\ lag=\frac{|\phi|}{360}\frac{1}{f}$$

$$phase\ lag=\frac{63.5\ cycle}{360}\frac{\pi\ \ min}{10\ cycle}$$

$$=0.0555min$$



A frequency of $\frac{10}{\pi}\frac{cycle}{min}$ means that a complete cycle occurs in $(\frac{10}{\pi})^{-1}$ min. since cycle is equivalent to 360^{o} and lag is 63.5^{o}

For a particular system for which the time constant τ is a fixed quantity, it is seen from Eq. (5.25) that the attenuation of amplitude and the phase angle ϕ depend only on the frequency ω. The attenuation and phase lag increase with frequency, but the phase lag can never exceed 90^{o} and approaches this value asymptotically.

The sinusoidal response is interpreted in terms of the mercury thermometer by the following example.

Example 5.2. A mercury thermometer having a time constant of 0.1 min is placed in a temperature bath at 100°F and allowed to come to equilibrium with the bath. At time t = 0, the temperature of the bath begins to vary sinusoidally about its average temperature of **100°F** with an amplitude of 2°F If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function of time. What is the phase lag?

In terms of the symbols used in this chapter

$$\tau = 0.1$$

$$x_s = 100°F$$

$$A = 2°F$$

$$f = \frac{10}{\pi}\ \text{cycles/min}$$

$$\omega = 2\pi f = 2\pi \frac{10}{\pi} = 20 \text{ rad/min}$$

From Eq. (5.25), the amplitude of the response and the phase angle are calculated; thus

$$\frac{A}{\sqrt{\tau^2\omega^2 + 1}} = \frac{2}{\sqrt{4 + 1}} = 0.896°F$$

$$\phi = -\tan^{-1} 2 = -63.5'$$

or

$$\text{Phase lag} = 63.5°$$

The response of the thermometer is therefore

$$Y(t) = 0.896 \sin (20t - 63.5°)$$

or

$$y(t) = 100 + 0.896 \sin (20t - 63.5°)$$

To obtain the lag in terms of time rather than angle, we proceed as follows: A frequency of $10/\pi$ cycles/min means that a complete cycle (peak' to peak) occurs in $(10/\pi)^{-1}$ min. Since one cycle is equivalent to 360" and the lag is 63.5°, the time corresponding to this lag is

$$\frac{63.5}{360} \times (\text{time for 1 cycle})$$

or

$$\text{Lag} = \frac{63.5}{360}\frac{\pi}{10} = 0.0555 \text{ min}$$

In general, the lag in units of time is given by

$$\text{Lag} = \frac{|\phi|}{360f}$$

when ϕ is expressed in degrees.

The response of the thermometer reading and the variation in bath temperature are shown in Fig. 5.8. It should be noted that the response shown in this figure holds only after sufficient time has elapsed for the nonperiodic term of Eq. (5.24) to become negligible. For all practical purposes this term becomes negligible after a time equal to about 37. If the response were desired beginning from the time the bath temperature begins to oscillate, it would be necessary to plot the complete response as given by Eq. (5.24).

Homework

Q1 state the Laplace transformation and sketch the input and first order response curve for input function given below

a-negative step of 20 units amplitude.

b-sinusoidal of 5 unit amplitude and 5 cycle /freq

c-ramp at 4 unit/sec delayed by 10 s.

d-step of 10 unit amplitude for 5 min than 10 units higher

e-impulse of 20 units amplitude returning to 10 units higher than the steady state .

Q2 state laplace transform and sketch both in put and first order system

a-sinusoidal of 8 units amp 4 cyc/sec

b-ramp at 8 unit delayed by 4 sec

Q3 state laplace transformation and sketch the curve

a-negative step of 20 units amp delayed by 10 sec

b- sinusoidal of 5 units amp and 5 sec period.

c-ramp at 4 units/sec for 10 sec than remains const at level reached

d- step of 20 units amplitude for 5min than 10 unit lower than that .

e-impulse of 20 units amplitude returning to level 10 units lower than the steady state.

f-negative impulse of 40 units returning to level lower the stady state .

Q4 state transfer function of the following disturbances and sketch the response first order ($k=2,\tau=10$) negative ramp at 10 units for 20 sec then positive ramp at 5 units/sec

Translation of function

Time Delay The most commonly used model to describe the dynamics of chemical process is First-Order Plus Model Delay Model. By proper choice τ_d, this model can be represent the dynamics of many industrial processes.

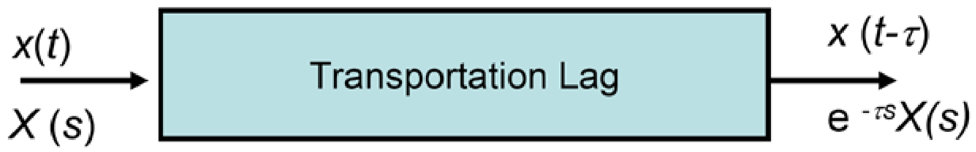
• Time delay or dead time between inputs and outputs are very common industrial procsses, engineering systems, economical, and biological systems.

• Transportation and measurement lags, analysis times, computation and communication lags.

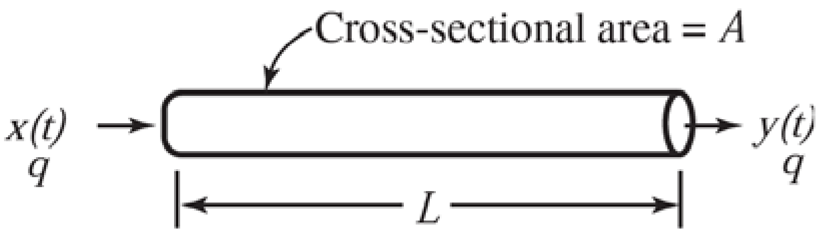
Any delay in measuring, in controller action, in actuator operation, in computer computation, and the like, is called transportation delay or dead time, and it always reduces the stability of a system and limits the achievable time of the system.

The Transportation Lag

The transportation lag is the delay between the time an input signal is applied to a system and the time the system reacts to that input signal. Transportation lags are common in industrial applications. They are often called “dead time”.



Dead-Time Approximations:-



$q_i(t)$ = Input to dead-time element. $q_o(t)$ = Output from dead-time element. The simplest dead-time approximation can be obtained qraphically or by physical representation.

The accuracy of this approxiamtion depends on the dead time being sufficicently small relative to the rate of the change of the slope of $q_i(t)$. If $q_i(t)$ were a ramp (constant slope), the approximation would be perfect for any value of τ_d. When the slope of $q_i(t)$ varies rapidly, only smal τ_d’s will give a good approximation. If the variation in x(t) were some arbitrary function, as shown in figure below, the response y(t) at the end of the pipe would be identical with x(t) but again delayed by t

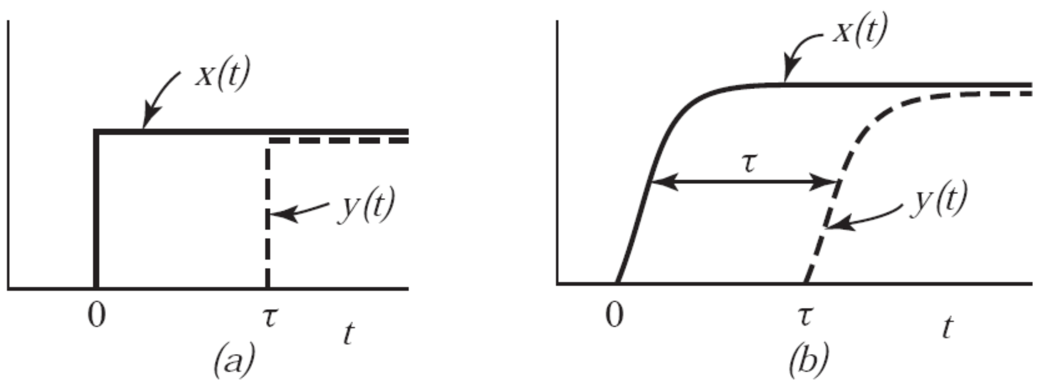
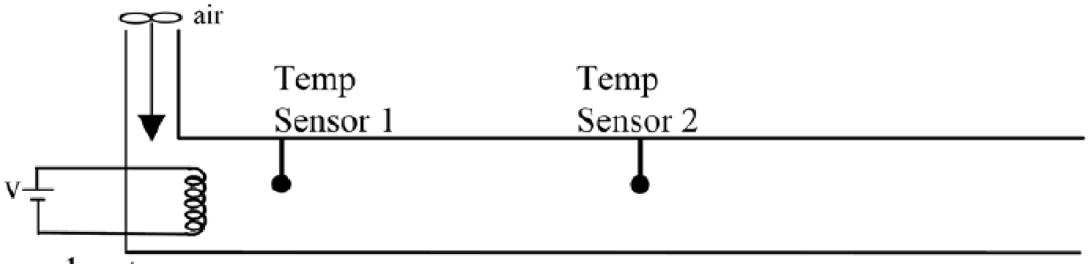


Figure Response of transportation lag to various inputs.

Example: **Thermal system**



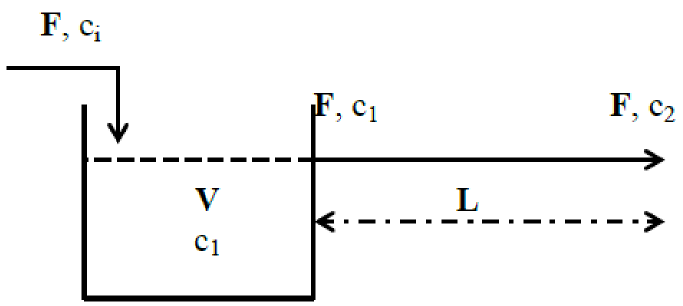
If measured at T_1 this can be modelled as:

$$\frac{T_1(s)}{V(s)} = \frac{K}{\tau s + 1}$$

Due to the delay time the temperature T_2 represented by:

$$\frac{T_2(s)}{V(s)} = \frac{K\,e^{-\tau_d s}}{\tau s + 1}$$

Example: **Mixing tank with time delay.**



F, V : Constants

Time delay

$$\frac{C_2(s)}{C_1(s)} = e^{-\tau_d s}$$

$$\tau_d = \frac{Volume\ of\ tube}{Volumetric\ flow\ rate} = \frac{AL}{q} = \frac{AL}{uA} = \frac{L}{u}$$

$$\frac{C_1(s)}{C_i(s)} = \frac{R}{\tau s + 1}$$

$$\because \frac{C_2(s)}{C_i(s)} = \frac{C_2(s)}{C_1(s)} \times \frac{C_1(s)}{C_i(s)}$$

$$\because \frac{C_2(s)}{C_i(s)} = \frac{C_2(s)}{C_1(s)} \times \frac{C_1(s)}{C_i(s)}$$

$$\therefore \boxed{\frac{C_2(s)}{C_i(s)} = \frac{R}{\tau s+1} e^{-\tau_d s}}$$ Where $$\boxed{\tau_d = \frac{L}{u}}$$ (Time Units)

SECOND-ORDER SYSTEM

Transfer Function

This section introduces a basic system called a *second-order system* or a *quadratic* lug. A second-order transfer function will be developed by considering a classical example from mechanics. This is the damped vibrator, which is shown in Fig. 8.1.

A block of mass *W* resting on a horizontal, frictionless table is attached to a linear spring. A viscous damper (dashpot) is also attached to the block. Assume that the system is free to oscillate horizontally under the influence of a forcing function F(t). The origin of the coordinate system is taken as the right edge of the block when the spring is in the relaxed or unstretched condition. At time zero, the block is assumed to be at rest at this origin. * Positive directions for force and displacement are indicated by the arrows in Fig. 8.1.

Consider the block at some instant when it is to the right of Y = 0 and when it is moving toward the right (positive direction). Under these conditions,

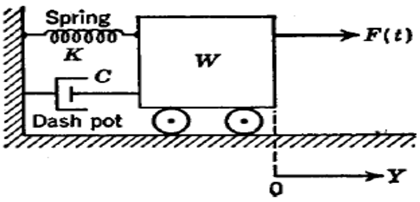


FIGURE 8-1
Damped vibrator.

the position Y and the velocity dY/dt arc both positive. At this particular instant, the following forces are acting on the block:

1. The force exerted by the spring (toward the left) of -KY where *K* is a positive constant, called Hooke's constant.
2. The viscous friction force (acting to the left) of $-C\, dY/dt$, where C is a positive constant called the damping coefficient.
3. The external force $F(t)$ (acting toward the right).

Newton's law of motion, which states that the sum of all forces acting on the mass is equal to the rate of change of momentum (mass × acceleration), takes the form

$$\frac{W}{g_c}\frac{d^2Y}{dt^2} = -KY - C\frac{dY}{dt} + F(t) \tag{8.1}$$

Rearrangement gives

$$\frac{W}{g_c}\frac{d^2Y}{dt^2} + C\frac{dY}{dt} + KY = F(t) \tag{8.2}$$

where W = mass of block, lb,

g_c = 32.2(lb$_m$)(ft)/(lb$_f$)(sec^2)

C = viscous damping coefficient, lb$_f$/(ft/sec)

K = Hooke's constant, lb$_f$/ft

$F(t)$ = driving force, a function of time, lb$_f$

Dividing Eq. (8.2) by K gives

$$\frac{W}{g_cK}\frac{d^2Y}{dt^2} + \frac{C}{K}\frac{dY}{dt} + Y = \frac{F(t)}{K} \quad (8.3)$$

For convenience, this is written as

$$\tau^2\frac{d^2Y}{dt^2} + 2\zeta\tau\frac{dY}{dt} + Y = X(t) \quad (8.4)$$

where

$$\tau^2 = \frac{W}{g_cK} \quad (8.5)$$

$$2\zeta\tau = \frac{C}{K} \quad (8.6)$$

$$X(t) = \frac{F(t)}{K} \quad (8.7)$$

Solving for τ and ζ from Eqs. (8.5) and (8.6) gives

$$\tau = \sqrt{\frac{W}{g_cK}} \quad \text{sec} \quad (8.8)$$

$$\zeta = \sqrt{\frac{g_cC^2}{4WK}} \quad \text{dimensionless} \quad (8.9)$$

By definition, both τ and ζ must be positive. The reason for introducing τ and ζ in the particular form shown in Eq. (8.4) will become clear when we discuss the solution of Eq. (8.4) for particular forcing functions $X(t)$.

Equation (8.4) is written in a standard form that is widely used in control theory. Notice that, because of superposition, $X(t)$ can be considered as a forcing function because it is proportional to the force $F(t)$.

If the block is motionless ($dY/dt = 0$) and located at its rest position ($Y = 0$) before the forcing function is applied, the Laplace transform of Eq. (8.4) becomes

$$\tau^2s^2Y(s) + 2\zeta\tau sY(s) + Y(s) = X(s) \quad (8.10)$$

From this, the transfer function follows:

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2s^2 + 2\zeta\tau s + 1} \quad (8.11)$$

The transfer function given by Eq. (8.11) is written in standard form, and we shall show later that other physical systems can be represented by a transfer function having the denominator $\tau^2s^2 + 2\zeta\tau s + 1$. All such systems are defined as second-order. Note that it requires two parameters, τ and ζ, to characterize the dynamics of a second-order system in contrast to only one parameter for a first-order system. For the time being, the variables and parameters of Eq. (8.11) can be interpreted in terms of the damped vibrator. We shall now discuss the response of a second-order system to some of the common forcing functions, namely, step, impulse, and sinusoidal.

$$G(s)=\frac{k\omega_n^2}{s^2+2\zeta\omega s+\omega_n^2}$$

ω un damped natural frequency

ζ the damping ratio

$$\omega_n = \frac{1}{\tau}$$

$$=\frac{-2\zeta\omega_n \pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2}$$

For ζ>1 over damped

ζ=1 critically damped

ζ<1 under damped

ζ=0 nodamped

Response of second order

1- Step change

$X(s)=\frac{A}{s}$

$Y(s)=\frac{k}{\tau^2 s^2+2\zeta s\tau+1}$

$=\frac{\frac{k}{\tau^2}}{s^2+\frac{2\zeta}{\tau}s+\frac{1}{\tau^2}}\cdot\frac{A}{s}$

$=\frac{\frac{-2\zeta}{\tau}\pm\sqrt{\left(\frac{2\zeta}{\tau}\right)^2-\frac{4}{\tau^2}}}{2}=\frac{-\zeta}{\tau}\pm\sqrt{\frac{\frac{4\zeta^2}{\tau^2}-\frac{4}{\tau}}{2}}=\frac{-\zeta}{\tau}\pm\frac{2\sqrt{\zeta^2-1}}{2\tau}=\frac{-\zeta}{\tau}\pm\frac{\sqrt{\zeta^2-1}}{\tau}$

$S1=\frac{-\zeta}{\tau}+\frac{\sqrt{\zeta^2-1}}{\tau}, s2=\frac{-\zeta}{\tau}-\frac{\sqrt{\zeta^2-1}}{\tau}$

$Y(s)=\frac{k/\tau^2}{s(s-s1)(s-s2)}$

$Y(s)=\frac{k}{\tau^2 s^2+2\zeta s\tau+1}\ \frac{A}{s}=\frac{\alpha_o}{s}+\frac{\alpha_1 s+\alpha_2}{\tau^2 s^2+2\zeta s\tau+1}$

$s^o\ \ \alpha_o=kA$

$s^1\qquad 2\zeta\tau+\alpha_2=0\ ,\ \alpha_2=-2kA\,\zeta\tau$

$s^2\qquad \alpha_o\tau^2+\alpha_1=0, \alpha_1=-kA\tau^2$

$Y(s)=kA\left[\frac{1}{s}-\frac{\tau^2 s+2\zeta\tau}{\tau^2 s^2+2\zeta s\tau+1}\right]$

$Y(s)= kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+2\frac{\zeta}{\tau}s+\frac{1}{\tau^2}\right)+\frac{1}{\tau^2}-\frac{\zeta^2}{\tau^2}}\right]=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}\right]$

1- For ζ<1 under damped system

$$=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]==kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$==kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$==kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}\cdot\frac{\tau}{\sqrt{1-\zeta^2}}\frac{\sqrt{1-\zeta^2}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$Y(t)=kA\left[1-e^{-\left(\frac{\zeta}{\tau}\right)t}cos\frac{\sqrt{1-\zeta^2}}{\tau}t-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\left(\frac{\zeta}{\tau}\right)t}\sin\frac{\sqrt{1-\zeta^2}}{\tau}\right]$$

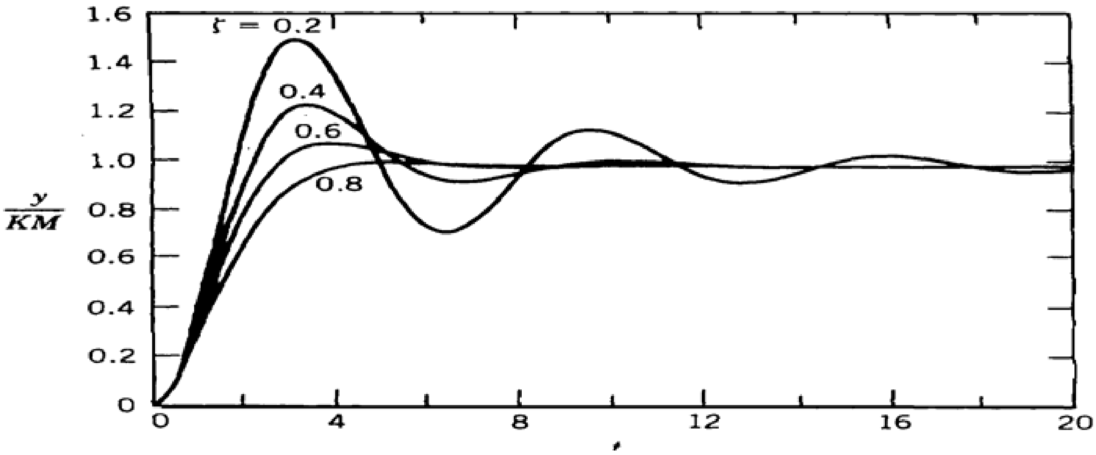
$$Y(t)=KA\left[1-e^{-\left(\frac{\zeta}{\tau}\right)t}\left(coswt+\frac{\zeta}{\sqrt{1-\zeta^2}}sinwt\right)\right],$$

$$W=\frac{\sqrt{1-\zeta^2}}{\tau}$$

$$r=\sqrt{p^2+q^2}=\sqrt{1+\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^2}=\sqrt{\frac{1}{\sqrt{1-\zeta^2}}}$$

$$\Theta=\tan^{-1}\frac{p}{q}=\tan^{-1}\frac{1}{\frac{\zeta}{\sqrt{1-\zeta^2}}}=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\tau}$$

$$Y(t)=kA\left[1-e^{-\left(\frac{\zeta}{\tau}\right)t}(rsinwt+\Theta)\right]$$



2- For over damped

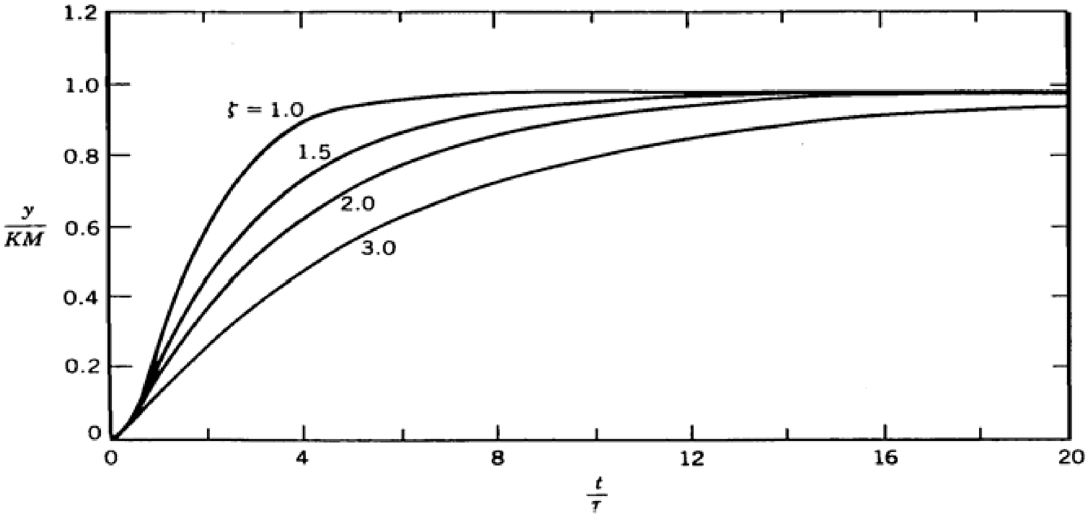
$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]=$$

$$= kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$= kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}\cdot\frac{\tau}{\sqrt{1-\zeta^2}}\frac{\sqrt{1-\zeta^2}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$Y(t)=KA\left[1-e^{-\left(\frac{\zeta}{\tau}\right)t}\left(coswt+\frac{\zeta}{\sqrt{1-\zeta^2}}sinwt\right)\right]$$

$$W=\frac{\sqrt{1-\zeta^2}}{\tau}$$



Terms Used to Describe an Underdamped System Second order system response for a step change

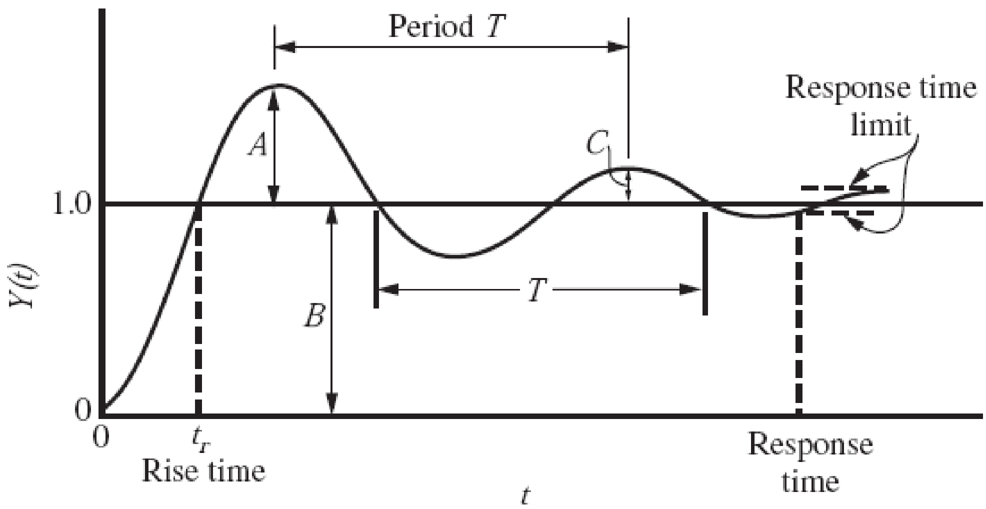


Figure (8.3) Terms used to describe an underdamped second-order response.

1. Overshoot(OS) Overshoot is a measure of how much the response exceeds the ultimate value (new steady-state value) following a step change and is expressed as the ratio $AABB$ in the Fig(8-3).

$$OS=\exp\frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

2. Decay ratio(DR) The decay ratio is defined as the ratio of the sizes of successive peaks and is given by $CCAA$ in Fig. (8.3). where C is the height of the second peak

DR=exp$\frac{-2\pi\xi}{\sqrt{1-\xi^2}}$=$(OS)^2$

3. Rise time (t_r) This is the time required for the response to first reach its ultimate value and is labeled in Fig. (8.3).$t_r=\frac{\pi-tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{w}$

4. Response time This is the time required for the response to come within ±5 percent of its ultimate value and remain there. The response time is indicated in Fig. (8.3).

5. Period of oscillation ***(T)*** The radian frequency (radians/time) is the coefficient of *t* in the sine term; thus,

$$T=\frac{2\pi\xi\tau}{\sqrt{1-\xi^2}}$$

6. Natural period of oscillation

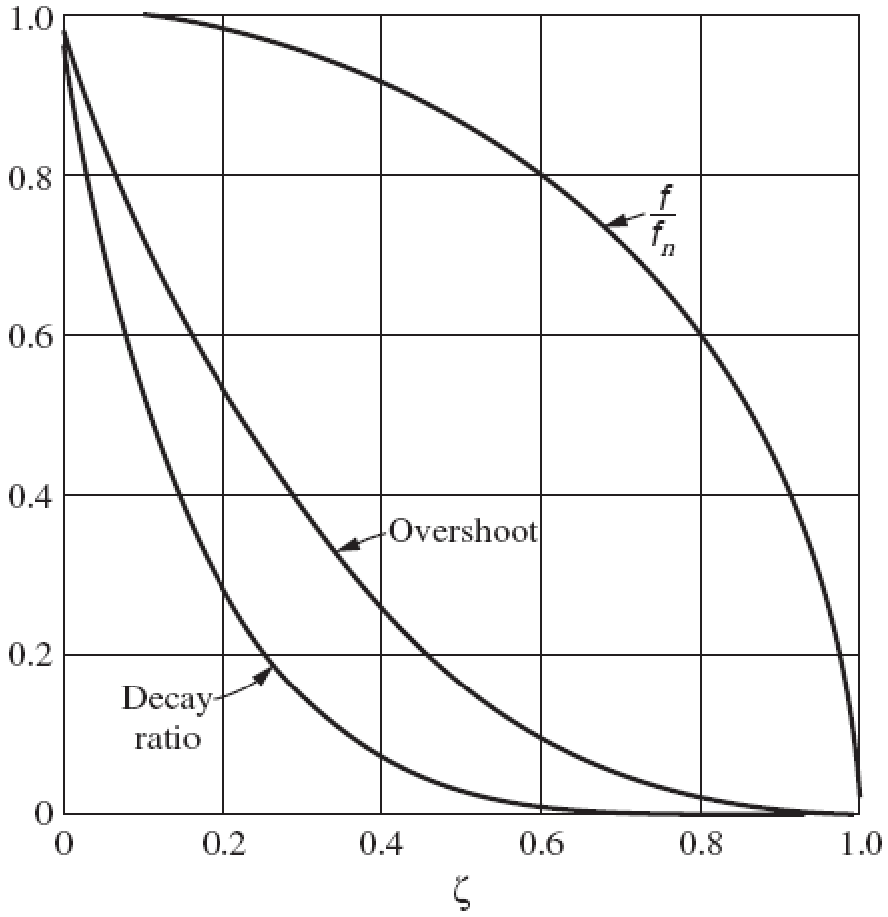
If the damping is eliminated (ξ=0), the system oscillates continuously without attenuation in amplitude. Under these "natural" or undamped condition, the radian frequency is . This frequency is referred to as the natural frequency $w_n=\frac{1}{\tau}$

$_n w$The corresponding natural cyclical frequency f_n and period T_n are related by the expression:-

$f_n=\frac{1}{T_n}=\frac{1}{2\pi\tau}$ Thus, τ has the significance of the undamped period.

7- Time to First Peak (t_p) **:** Is the time required for the output to reach its first maximum value.

$$t_p=\frac{\pi}{w}=\frac{\pi\tau}{\sqrt{1-\xi^2}}$$



Figure(8.4) Characteristics of a step response of underdamped second-order system

Notes:for the step response
1-over damped
Very slowly rise time so long rise time but long settling time
2-under damped
Very long settling time but the rise time is short
3-un damped
Very long rise time but there is no settling time
4-critically damped
Short rise time short settling time.

Derivations

1-over shoot

wt+Θ=Θ+nπ

t=$\frac{n\pi}{w}$ max or min n=1,2,3

if n=0,2,4,6 min

if n=1,3,5,7 max

first max when n=1

t=$\frac{n\pi}{w} = \frac{\pi}{w}$

Y(t)=kA$\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta\pi}{\tau w}} \sin(w\frac{\pi}{w} + \Theta)\right]$

$$Ymax=kA\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}(-\sin\Theta)\right]$$

For under damped cos Θ=-ζ,sinΘ=$\sqrt{1-\zeta^2}$

Tan Θ=$\frac{\sqrt{1-\zeta^2}}{-\zeta}$

$$Ymax=KA\left[1+\frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\sqrt{1-\zeta^2}\right]$$

$$Ymax=KA\left[1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

Over shoot=$\frac{A}{B}=\frac{max-B}{B}$

Over shoot=$\frac{kA\left[1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]-kA}{kA}$,over shoot=exp$\left[1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$

2-Decay Ratio

Decay ratio=$\frac{C}{A}$(the ratio of amount above the ultimate value of two successive peaks), t=$\frac{n\pi}{w}$for n=3 then t=$\frac{3\pi}{w}$

First peak at n=1 Ymax=KA$\left[1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$

Second peak n=3 Ymax=KA$\left[1+e^{-\frac{3\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$

Decay ratio=$\frac{KA\left[1+e^{-\frac{3\zeta\pi}{\sqrt{1-\zeta^2}}}\right]-kA}{KA\left[1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]-kA}=\frac{e^{-\frac{3\zeta\pi}{\sqrt{1-\zeta^2}}}}{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}=e^{-\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}}=exp\frac{-2\zeta\pi}{\sqrt{1-\zeta^2}}$

*U3. **Rise timeU** (tRrR)* It is the time required for the response to first tauch the ultimate line.

$$Y(t)=kA\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\zeta t}{\tau}}\sin(tw+\Theta)\right]$$

At tr Y(t)=kA

$$kA=kA\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\zeta t}{\tau}}\sin(t_r w+\Theta)\right]$$

$$0=\sin(t_r w+\Theta),\ t_r=\frac{\sin^{-1}(0)-\Theta}{w},\ t_r=\frac{n\pi-\Theta}{w}=\frac{n\pi-\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{w}$$

$$t_r=\frac{\pi-\tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta}}{w}\quad \text{for n=1}$$

4-period of oscillation

W=radian frequency $=\frac{\sqrt{1-\zeta^2}}{\tau}$,w=$2\pi f$,T=$\frac{1}{f}$

F= $\frac{\sqrt{1-\zeta^2}}{2\pi\tau}$

T=$\frac{2\pi\tau}{\sqrt{1-\zeta^2}}$

***5. Natural period of oscillation** (TRnR).* The system free of any damping for ξ=0,w radian of frequency=$\frac{\sqrt{1-\zeta^2}}{\tau}$,$w_n=\frac{1}{\tau}$ for ζ=0, $w_n=2\pi f_n$,$\frac{1}{\tau}=2\pi f_n$

$$f_n=\frac{1}{2\pi\tau}$$

6-Response time t_s: The time required for the response to reach (±5%) of its ultimate value and remain there.

***7- Time to First Peak** (tRpR)* Is the time required for the output to reach its first maximum value.t=$\frac{n\pi}{w}$

First peak is reached when n=1, ***tRpR***=$\frac{n\pi}{w}=\frac{\pi}{w}=\frac{\pi\tau}{\sqrt{1-\zeta^2}}$

3- Impulse Response

If impulse δ(t) is applied to second order system then transfer response can be written

$$Y(s)=\frac{k}{\tau^2 s^2+2\zeta s\tau+1}$$

X(s)=A=area

$$Y(s)=\frac{k}{\tau^2 s^2+2\zeta s\tau+1}.A$$

$$Y(s)=\frac{kA/\tau^2}{s^2+2\frac{\zeta}{\tau}s+\frac{1}{\tau^2}}=\frac{kA/\tau^2}{s^2+2\frac{\zeta}{\tau}s+\frac{1}{\tau^2}+\left(\frac{\zeta}{\tau}\right)^2-\left(\frac{\zeta}{\tau}\right)^2}=\frac{kA/\tau^2}{s^2+2\frac{\zeta}{\tau}s+\left(\frac{\zeta}{\tau}\right)^2+\frac{1}{\tau^2}-\left(\frac{\zeta}{\tau}\right)^2}=\frac{kA/\tau^2}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}$$

i-ζ>1

$$Y(s)=\frac{\frac{kA}{\tau^2}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}=\frac{\frac{kA}{\tau^2}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{1-\zeta^2}}\frac{\sqrt{1-\zeta^2}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}$$

$$=\frac{kA}{\tau\sqrt{\zeta^2-1}}e^{-\frac{\zeta t}{\tau}}sinhwt, w=\frac{\sqrt{\zeta^2-1}}{\tau}$$

ii-ζ<1

$$y(s)=\frac{kA/\tau^2}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}$$

$$=\frac{\frac{kA}{\tau^2}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}=\frac{kA}{\tau\sqrt{1-\zeta^2}}e^{-\frac{\zeta t}{\tau}}sinwt, w=\frac{\sqrt{1-\zeta^2}}{\tau}$$

iii-ζ=1

$$Y(s)=\frac{\frac{kA}{\tau^2}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}=\frac{\frac{kA}{\tau^2}}{\left(s+\frac{1}{\tau}\right)^2+\frac{1-1}{\tau^2}}=\frac{kA}{\tau^2}te^{-\frac{t}{\tau}}$$

Example A step change from 15 to 31 psi in actual pressure results in the measured response from a pressure indicating element shown in Fig. E5.14.

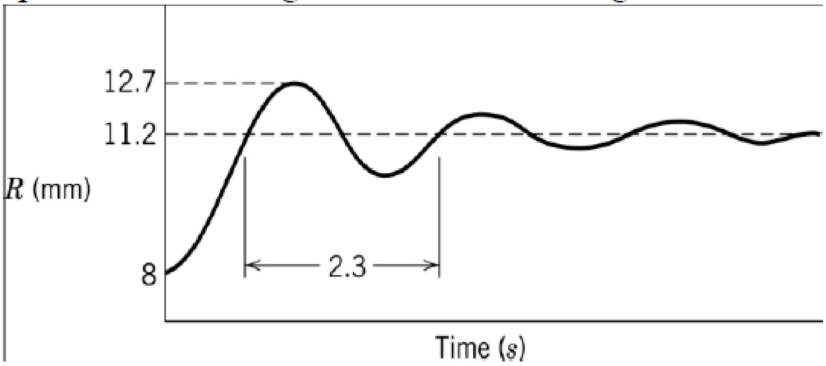


Figure E5.14

Assuming second-order dynamics, calculate all important parameters and write and approximate transfer function in the form

$$\frac{R'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where R' is the instrument output deviation (mm), P' is the actual pressure deviation (psi).

Solution:

$$Gain = \frac{11.2\,mm - 8\,mm}{31\,psi - 15\,psi} = 0.20\ mm/psi$$

$$Overshoot = \frac{12.7\,mm - 11.2\,mm}{11.2\,psi - 8\,psi} = 0.47$$

$$Overshoot = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.47$$

$$\zeta = 0.234$$

$$Period = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} = 2.3\sec$$

$$\tau = 2.3\sec \times \frac{\sqrt{1-0.234^2}}{2\pi} = 0.356\sec$$

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.127s^2 + 0.167s + 1}$$

Example: A control system having transfer function is expressed as:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{5}{\tau^2 s^2 + 2\psi\tau s + 1}$$

The radian frequency for the control system is 1.9 rad/min. The time constant is 0.5 min. The control system is subjected to a step change of the magnitude 2.

Calculate :

(i) Rise time

(ii) Decay ratio

(iii) Maximum value of Y(t)

(iv) Response time

Solution:

Given

$$X(s) = \frac{2}{s}$$

Time constant $\tau = 0.5\ \text{min}$

Radian frequency $w = 1.9\ rad/\text{min}$

$$w = \frac{\sqrt{1-\xi^2}}{\tau} \quad \Rightarrow \quad 1.9 = \frac{\sqrt{1-\xi^2}}{0.5} \quad \Rightarrow \quad \xi = 0.312$$

i) Rise time

$$tr = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{w} = \frac{3.1416 - \tan^{-1}\frac{\sqrt{1-0.312^2}}{0.312}}{1.9} = 1.0\ \text{min}$$

ii) Deacay ratio $= \frac{C}{A} = \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right) = \exp\left(\frac{-2\pi 0.312}{\sqrt{1-0.312^2}}\right)$

$$\therefore Decay\ ratio = 0.127$$

iii) Ultimate value of the response $Y_{ultimate}$ (B) at $t \to \infty$

$$\frac{Y(s)}{X(s)} = \frac{5}{0.25s^2 + 0.316s + 1}$$

$$X(s) = \frac{2}{s}$$

$$Y(s) = \frac{10}{s(0.25s^2 + 0.316s + 1)}$$

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0} \frac{10}{(0.25s^2 + 0.316s + 1)} = 10$$

$$\mathrm{Y_{ultimate}}\,(\mathrm{B}) = 10$$

$$\text{Maximum value of response } = B(1 + \frac{B}{A})$$

$$\text{Overshoot} = \frac{B}{A} = \exp(\frac{-\pi\xi}{\sqrt{1-\xi^2}})$$

$$\text{Decay ratio} = \text{Overshoot}^2$$

$$0.127 = \text{Overshoot}^2$$

$$\text{overshoot} = 0.356 = \frac{B}{A}$$

$$\text{Maximum value of response} = 10(1+0.356) = 13.56$$

iv) Response time $ts = 3\frac{\tau}{\xi} = 4.8077$ min for $\pm 5\%$ *of ultimate value*

The Control System

The control system A liquid stream at a temperature T_i, enters an insulated, well-stirred tank at a constant flow rate w (mass/time). It is desired to maintain (or control) the temperature in the tank at T_R by means of the controller. If the indicated (measured) tank temperature T_m differs from the desired temperature T_R, the controller senses the difference or ***error,*** $E = T_R - T_m$

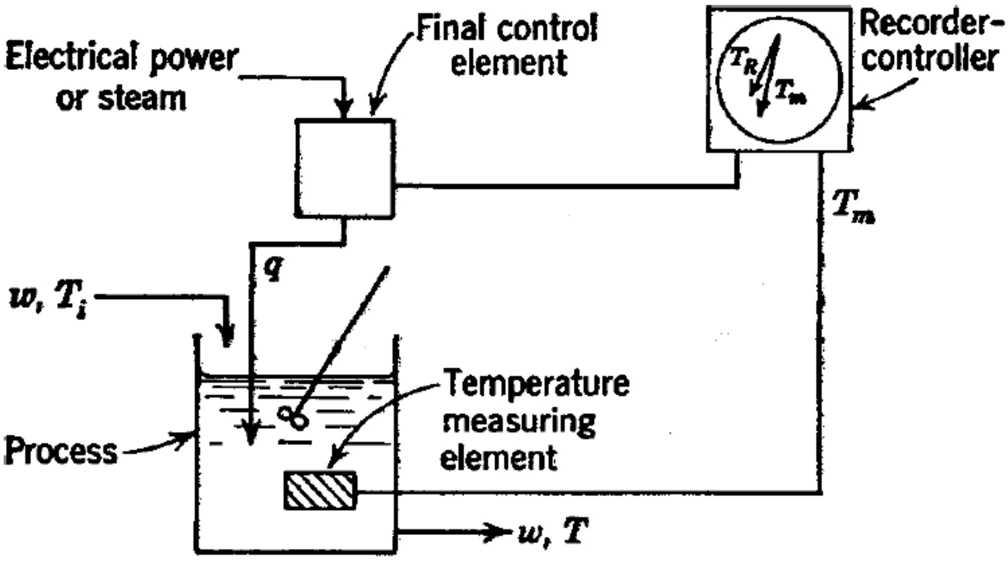


Figure (8.1) Control system for a stirred-tank heater.

There are two types of the control system:-

1) **Negative feedback control system**

Negative feedback ensures that the difference between T_R and T_m is used to adjust the control element so that tendency is to reduce the error. $E=T_R-T_m$

2) **Positve feedback control system**

If the signal to the compartos were obtained by adding T_R and T_m we would have a positive feedback systems which is inherently unstable. To see that this is true, again assume that be system is at steady state and that $T=T_R=T_i$. If T_i were to increase, T and T_m would increas which would cause the signal from the compartor to increase, with the result that the heat to the system would increse. At s.s. $T=T_R=T_{in}$ $E=T_R+T_m$

Servo Problem versus Requlator Problem

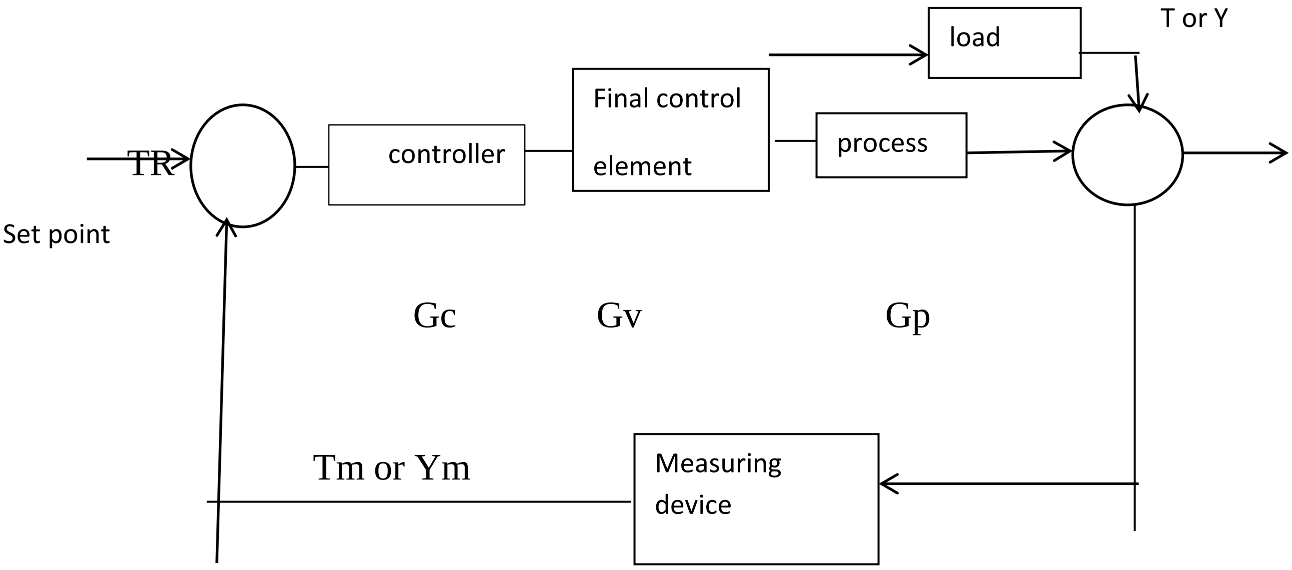
- **Servo Problem**

There is no change in load T_i, and that we are interested in changing the bath temperature (change in the desired value (set point) with no disturbance load).

- **Requlating problem**

The desired value T_R is to remain fixed and the purpose of the control system is to maintain the controlled variable T_R in spite of change in load if there is a change in the input variable (disturbance load). **Control system elements** Control system elements are:-

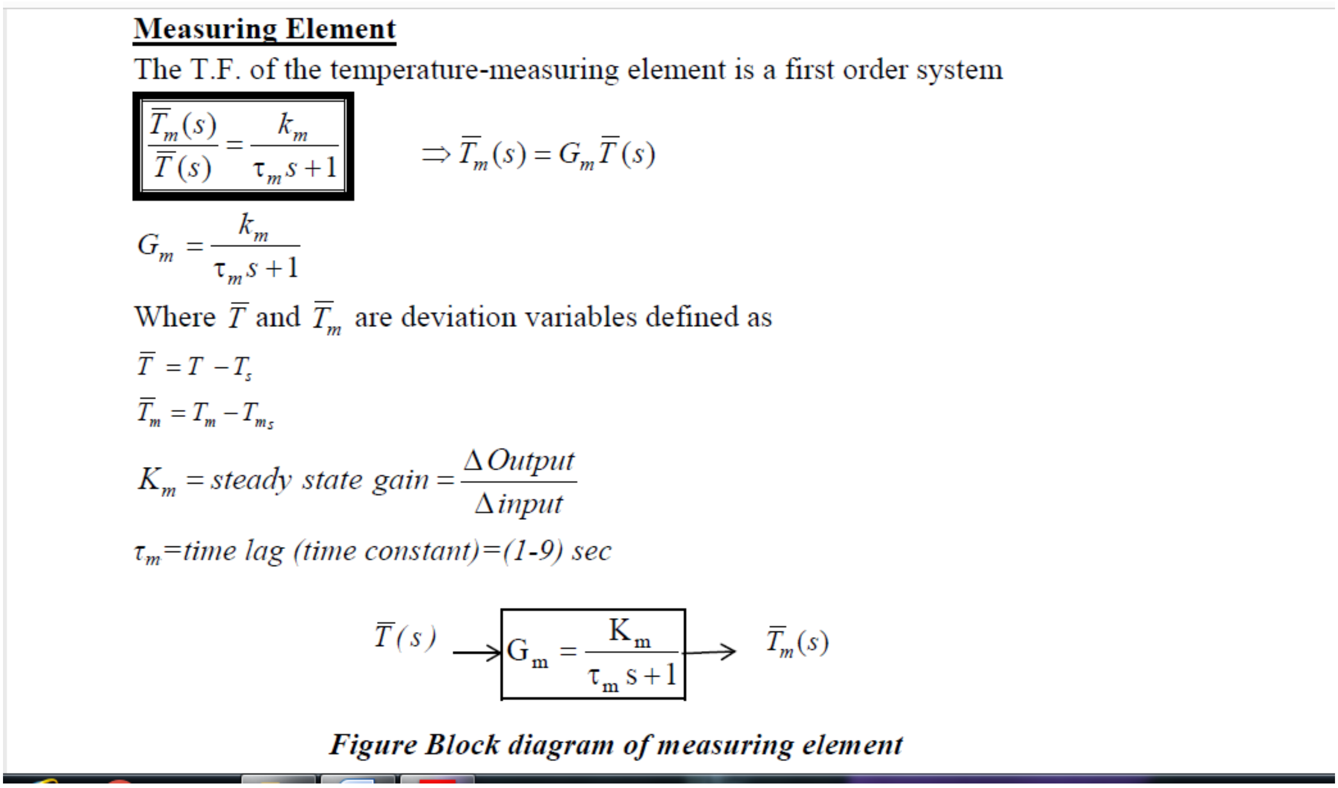
1) Process
2) Measuring element
3) Controller
4) Final Control Element



Closed Loop Feedback control

Measuring Element

The T.F. of the temperature-measuring element is a first order system



Measuring Element

The T.F. of the temperature-measuring element is a first order system

$$\frac{\bar{T}_m(s)}{\bar{T}(s)} = \frac{k_m}{\tau_m s+1} \qquad \Rightarrow \bar{T}_m(s) = G_m \bar{T}(s)$$

$$G_m = \frac{k_m}{\tau_m s+1}$$

Where $\bar{T}$ and $\bar{T}_m$ are deviation variables defined as

$$\bar{T} = T - T_s$$

$$\bar{T}_m = T_m - T_{m_s}$$

$$K_m = steady\ state\ gain = \frac{\Delta Output}{\Delta input}$$

τ_m=*time lag (time constant)=(1-9) sec*

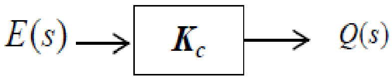
$$\bar{T}(s) \longrightarrow \boxed{G_m = \frac{K_m}{\tau_m s+1}} \longrightarrow \bar{T}_m(s)$$

Figure Block diagram of measuring element

Controller and final control element

The relationship for proportional controller is

$$\frac{P(s)}{G(s)} = G_c(s)$$



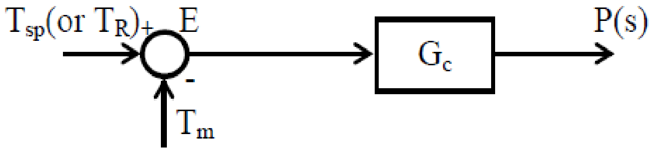
$$Q(s) = K_C E(s)$$

$$P = \overline{P} - \overline{P}_s$$

$$E = \overline{T}_R - \overline{T}_m$$

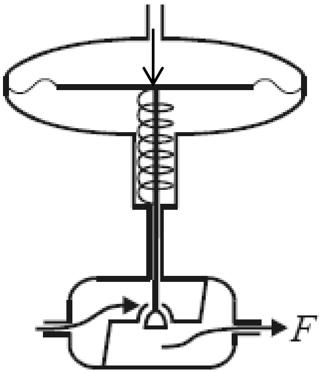
G(s) for propertional controller $G_c(s) = K_C$

$\overline{T}_R = \overline{T}_m = \overline{T}$ at steady state



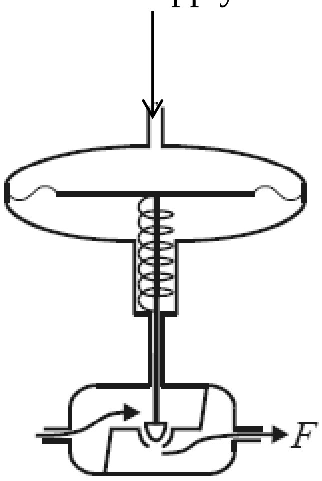
Controllers and Final Control Element

Final control Elements: Control valve, Heater, Variec, Motor, pump, damper, louver, …. etc. **Control valve** Control valve that can control the rate of flow of a fluid in proportion to the amplitude of a pressure (electrical) signal from the controller. From experiments conducted on pneumatic valves, the relationship between flow and valve-top pressure for a linear valve can often be represented by a first-order transfer function: Air supply



Control valve (Air to close)

Air supply



Control valve (Air to open)

(air to close)
open close
↓ ↓
$P = (3-15)$ psig
↓ ↓
max flow rate | flow rate $= 0$

(air to open)
close open
↓ ↓
$P = (3-15)$ psig
↓ ↓
flow rate $= 0$ | max flow rate

Transfer Function of Control Valve

$$Gv(s) = \frac{m(s)}{p(s)} = \frac{Q(s)}{P(s)} = \frac{K_V}{\tau_v s + 1}$$

$$K_v\text{=steady state gain} = \frac{\Delta\ \text{Output}}{\Delta\ \text{Input}} = \left(\frac{Q_2 - Q_1}{P_2 - P_1}\right)_{S.S}$$

$\tau_v = \text{Time lag}$

$\tau_v \le 10\ \text{sec}$ (Good)

Where:

K_v: steady-state gain i.e., the constant of proportionality between steady-state flow rate and valve-top pressure. τ_v: time constant of the valve and is very small compared with the time constants of other components of the control system. A typical pneumatic valve has a time constant of the order of 1 sec. Many industrial processes behave as first-order systems or as a series of first-order systems having time constants that may range from a minute to an hour. So the lag of the valve is negligible and the T. F. of the valve sometimes is approximated by: $k_v = \frac{Q(s)}{P(s)}$

The time constant of lag valve depends on the size of valve, air supply characteristics, whether a valve positioner is used, etc. **Control Action** It is the manner, in which the automatic controller compares the actual value of the process output with the actual desired value, determines the deviations and produce a control signal which will reduce the deviation to zero or to small value. **Classification of industrial automatic controller:** They are classified according to their control action as:

1) On-off controller
2) Proportional controller (P)
3) Integral controller (I)
4) Proportional plus Integral controller (PI)
5) Proportional plus Derivative controller (PD)
6) Proportional plus Integral plus Derivative controller (PID)

The automatic controller may be classified according to the kind of power employed in the operation, such as pneumatic controller, hydraulic controller or electronic controller.

Self-operated controller: In this controller the measuring element (sensor) and the actuator in one unit. It is widely used for the water and gas pressure control.

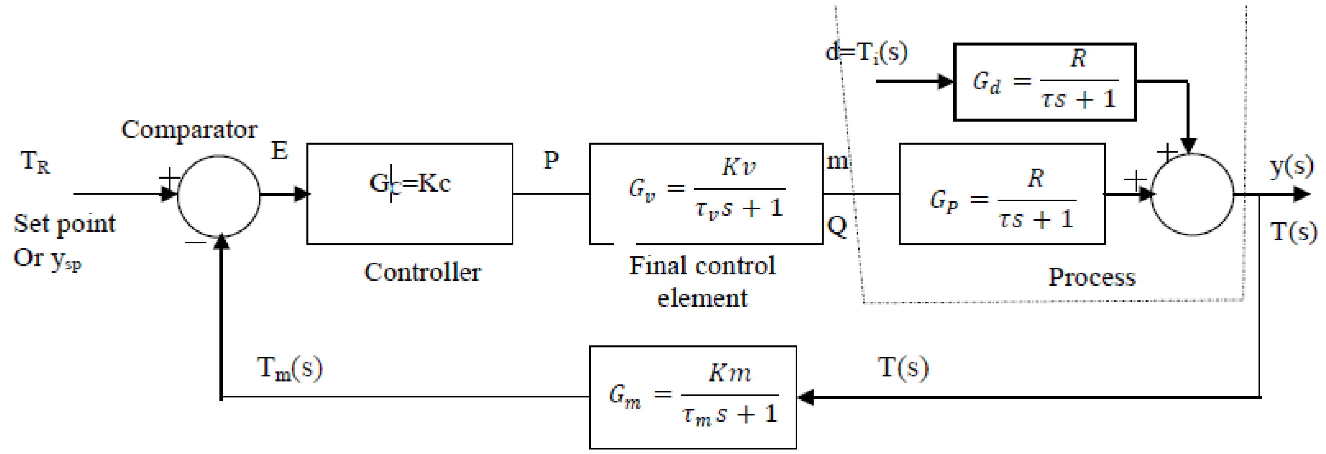


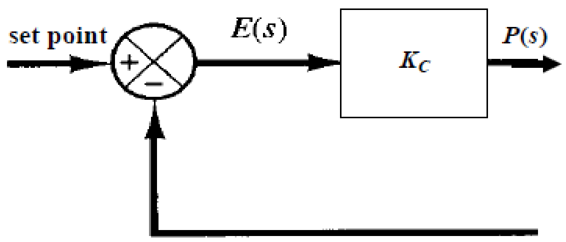
Figure: Closed loop block diagram of first order system

Types of Feedback Controllers 1) Proportional controller (P): For a controller with a proportional control action, the relationship between the output of the controller, p(t), and the actuating error signal (input to controller) is

$p \alpha \epsilon(t)$

p(t)=$k_c\epsilon(t) + p_s, p(t) - p_s = k_c\epsilon(t), p(s) = k_c\epsilon(s)$

$G_c = k_c = \frac{p(s)}{\epsilon(s)}$



Proportional Band (Band Width) Is defined as the error (expressed as a percentage of the range of measured variable) required to make the valve from fully close to fully open.P.B=$\frac{1}{k_c}100$

On-Off Control On-Off control is a special case of proportional control. If the gain Kc is made very high, the valve will move from one extreme position to the other if the set point is slightly changed. So the valve is either fully open or fully closed (The valve acts like a switch). The P.B. of the on-off controller reaches a zero because the gain is very high P.B=0

2) Propertional-Integral controller (PI): This mode of control is described by the relationship

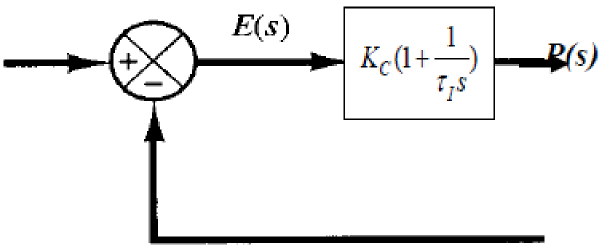
$$P(t)=k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt + p_s$$

$$P(t)-p_s = k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$$

$$P= k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$$

$$p(s)= k_c\epsilon(s)+\frac{k}{\tau_I}\frac{\epsilon(s)}{s},$$

$$G_c = \frac{p(s)}{\epsilon(s)},\ G_c = k_c + \frac{k_c}{\tau_I s},\ G_c = k_c(1+\frac{1}{\tau_I s})$$



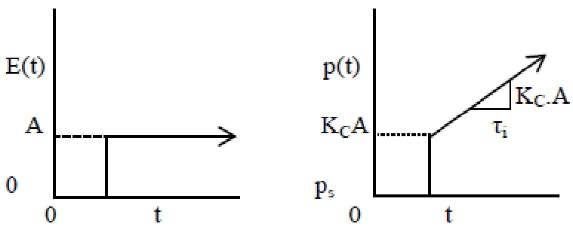
Prob(10-1): PI controller with step change in error $E(s)=\frac{A}{s}$

$$P(s)=K_C(1+\frac{1}{\tau_I s})\frac{A}{s}$$

$$\therefore P(t)=K_C A+\frac{K_C A}{\tau_I}t$$

Y=c+mX

t	E(t)	P(t)



Response of a PI controller (lineaer)

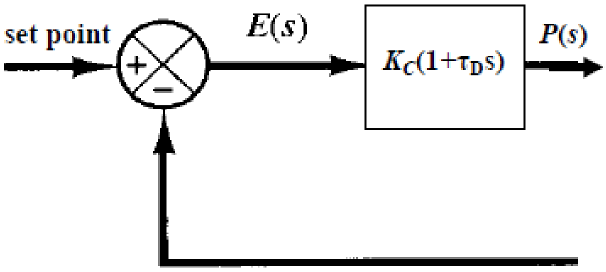
3) Proportional-derivative control (PD):

$$P(t)=k_c\epsilon(t) + k_c\tau_D\frac{d\epsilon(t)}{dt} + p_s$$

$$p(t)-p_s = k_c\epsilon(t) + k_c\tau_D\frac{d\epsilon(t)}{dt}$$

$$p(t)-ps= k_c\epsilon(t) + k_c\tau_D\frac{d\epsilon(t)}{dt}$$

p(s)= $k_c\epsilon(s) + k_c\tau_D s\epsilon(s)$, $G_c = \frac{p(s)}{\epsilon(s)} = k_c + k_c\tau_D s = k_c(1 + \tau_D s)$



K_C: gain τ_D : Derivative time (rate time)

Example:

For Ramp Error $\quad E(t) = At \quad$ (Ramp) $\quad E(s) = \frac{A}{s^2}$

$$P(s) = K_C(1+\tau_D s) \times E(s) = K_C(1+\tau_D s) \times \frac{A}{s^2} = \frac{AK_C}{s^2} + \frac{K_C A\tau_D}{s}$$

$$P(t) = K_C At + K_C A\tau_D$$

4) Proportional-Integral-Derivative (PID) controller

$$P(t)=k_c\epsilon(t) + \frac{k}{\tau_I}\int_0^t \epsilon(t)dt + k_c\tau_D \frac{d\epsilon(t)}{dt} + p_s$$

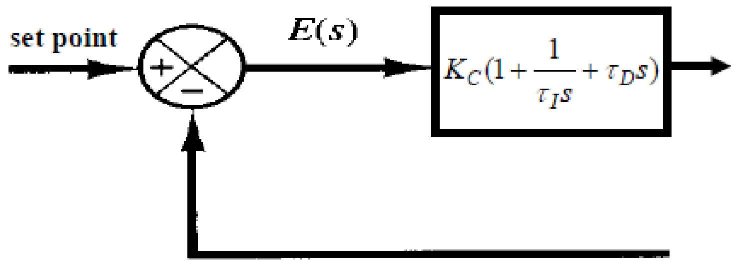
$$P(t)-p_s = k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt+k_c\tau_D \frac{d\epsilon(t)}{dt}$$

$$P= k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt+k_c\tau_D \frac{d\epsilon(t)}{dt}$$

$$p(s)= k_c\epsilon(s)+\frac{k}{\tau_I}\frac{\epsilon(s)}{s}+k_c\tau_D s\epsilon(s),$$

$$G_c = \frac{p(s)}{\epsilon(s)} = k_c + \frac{k_c}{\tau_I s} + k_c\tau_D s=k_c(1 + \frac{1}{\tau_I s} + \tau_D s)$$

ontroller



Motivation for Addition of Integral and Derivative Control Modes The value of the controlled variable is seen to rise at time zero owing to the disturbance. With no control, this variable continues to rise to a new steady-state value.

- With control, after some time the control system begins to take action to try to maintain the controlled variable close to the value that existed before the disturbance occurred.

- With proportional action only, the control system is able to arrest the rise of the controlled variable and ultimately bring it to rest at a new steady-state value. The difference between this new steady-state value and the original value (the set point, in this case) is called *offset.*

- The addition of integral action eliminates the offset; the controlled variable ultimately returns to the original value. This advantage of integral action is balanced by the disadvantage of a more oscillatory behavior.

- The addition of derivative action to the PI action gives a definite improvement in the response. The rise of the controlled variable is arrested more quickly, and it is returned rapidly to the original value with little or no oscillation.

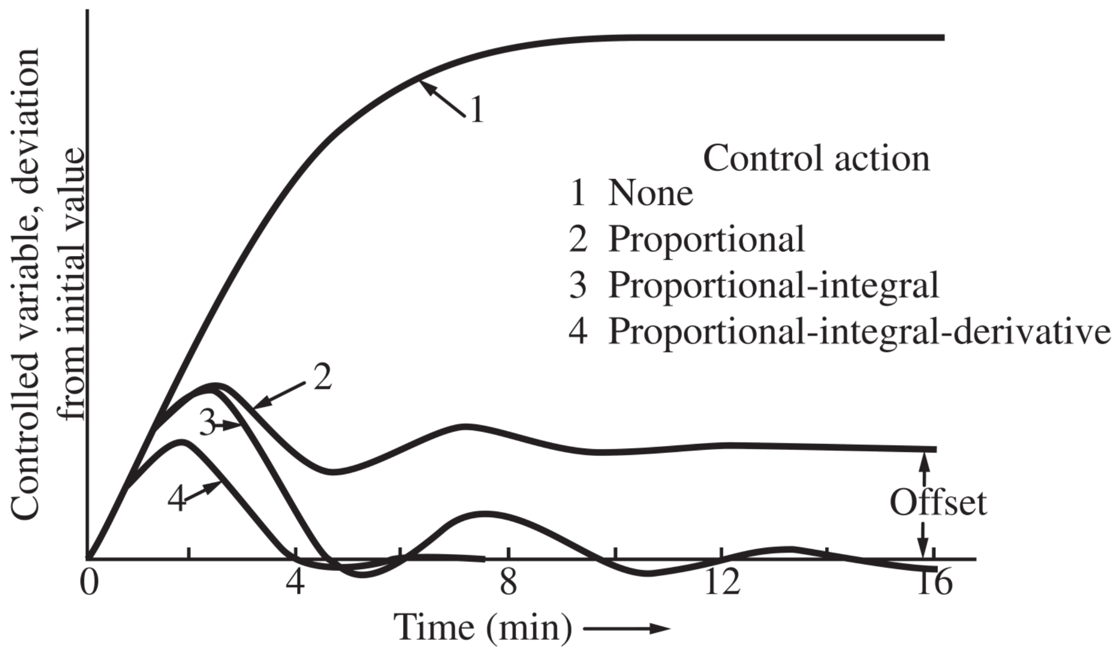


Figure: Response of a typical control system showing the effects of various modes of control

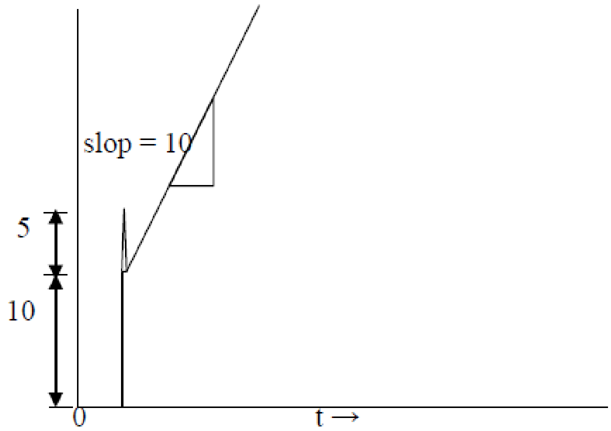
Example: A unit-step change in error is introduced into a PID controller. If K_C = 10, τ_I = 1, and τ_D = 0.5, plot the response of the controller, m(t).

Solution: The equation of PID controller is

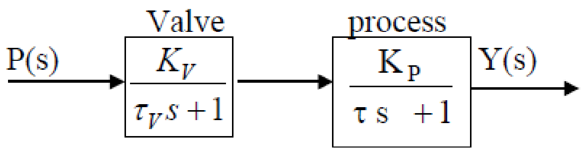
$$G_c = \frac{p(s)}{\epsilon(s)} = k_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \text{m(t)}.$$

$\mathbf{E(s)=\frac{1}{s}, P(s)=}\frac{10}{s}(1+\frac{1}{s} + 0.5s) = \frac{10}{s} + \frac{10}{s^2} + 5$

m(t)=10+10t+5δ(t)



Example: Consider the 1st order T. F. of the process with control valve



If we assume no interaction; The T. F. from P(s) to Y(s) is

$$\frac{Y(s)}{P(s)} = \frac{K_V K_p}{(\tau_v s + 1)(\tau\ s + 1)}$$ For a unit step input in P

$$\text{Y(s)} = \frac{1}{s}\frac{K_V K_p}{(\tau_v s + 1)(\tau s + 1)}$$

$$\text{y(t)} = K_v K_p\left[1 - \frac{\tau_v \tau}{\tau_v - \tau}\left(\frac{1}{\tau}e^{-t/\tau_v} - \frac{1}{\tau_v}e^{-t/\tau}\right)\right]$$

If $\tau >> \tau_v$ then the T. F. is $\frac{Y(s)}{P(s)} = \frac{K_v K_p}{(\tau s + 1)}$

For a unit step input in p

y(t) = $K_v K_p (1 - e^{-t/\tau})$

Example: a pneumatic PI controller has an output pressure of 10 psi when the set point and pen point are togather. The set point is suddenly displaced by 1.0 in (i.e a step change in error is introduced) and the follwing data are obtained.

Time (s) 0 0 20 60 80

Psi 10 8 7 5 3.5

Determine the actual gain (psi/inch displacement) and the integral time

P=$k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt + p_s$

p-$p_s = k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$

p(t)= $k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$

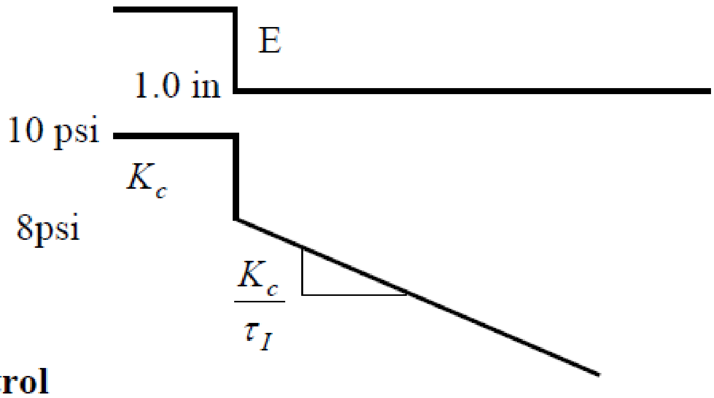
p(s)= $k_c\epsilon(s)+\frac{k}{\tau_I}\frac{\epsilon(s)}{s}$,

$G_c = \frac{p(s)}{\epsilon(s)}, G_c = k_c + \frac{k_c}{\tau_I s}, G_c = k_c(1+\frac{1}{\tau_I s})$

$k_c = 2$

$$\frac{k_c}{\tau_I} = \frac{7-5}{60-20} = \frac{2}{40} =$$

τ_I = 20x2=40 sec



For PI control

Example: (A) a unit-step change in error is introduced into a pid controller, if K_c=10 , τ_I=1 and τ_D=0.5 plot the response of the controller P(t). (B) if the error changed with a ratio of 0.5 in/min plot the response of p(t).

a-p=$k_c\epsilon(t)+\frac{k}{\tau_I}\int_0^t \epsilon(t)dt+k_c\tau_D\frac{d\epsilon(t)}{dt}+p_s$

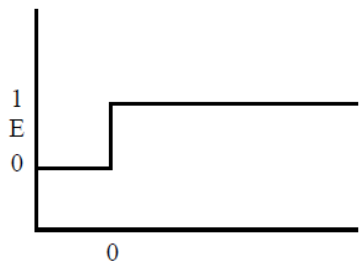
p-$p_s=k_c\epsilon(t)$+$\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$+$k_c\tau_D\frac{d\epsilon(t)}{dt}$

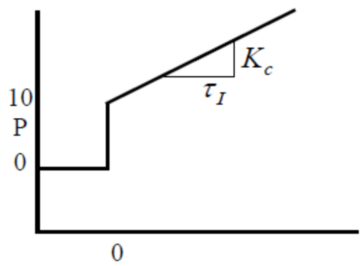
p(t)= $k_c\epsilon(t)$+$\frac{k}{\tau_I}\int_0^t \epsilon(t)dt$+$k_c\tau_D\frac{d\epsilon(t)}{dt}$

p(s)= $k_c\epsilon(s)$+$\frac{k}{\tau_I}\frac{\epsilon(s)}{s}$+$k_c\tau_D s\epsilon(s)$,

$G_c=\frac{p(s)}{\epsilon(s)}=k_c+\frac{k_c}{\tau_I s}+k_c\tau_D s$=$k_c(1+\frac{1}{\tau_I s}+\tau_D s)$

P(t)=10+10t

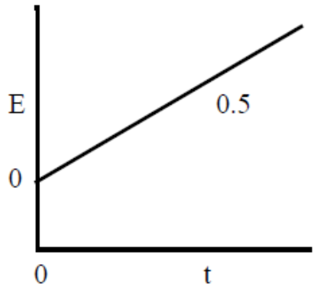


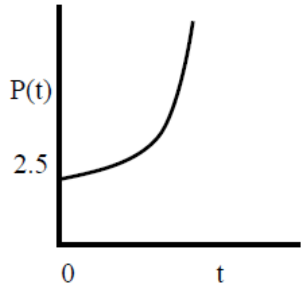


b-E=0.5t,$\frac{dE}{dt}=0.5,\int dE\ dt=\int 0.5dt$

p(t)=10x0.5t+10$\int 0.5tdt$+10x0.5x0.5=2.5+5t+2.5 t^2

$= p(t) - p_s = 2.5 + 5t + 2.5t$

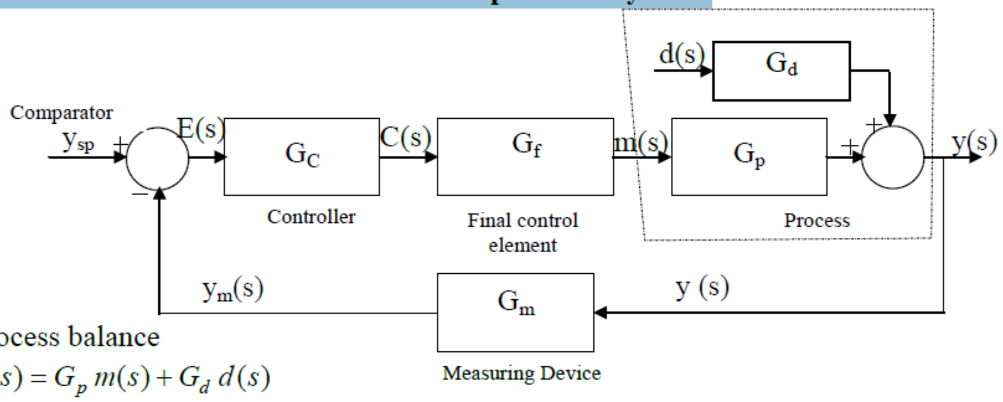




t	P(t)
0	2.5
1	10
2	22.5
3	40
4	62.5
5	90

Dynamic Behaviour of Feedback Controlled Process

Overall transfer function of a closed- loop control system:



Process balance

$$y(s) = G_p\, m(s) + G_d\, d(s)$$

Measuring device

$$y_m(s) = G_m\, y(s)$$

Controller system

Controller system

$E(s) = y_{sp}(s) - y_m(s)$ Comparator

$C(s) = G_c E(s)$ Controller

Final control element

$m(s) = G_f C(s)$

Algebra manipulation of the above equations and arrange then

$$y(s) = G_p\, m(s) + G_d\, d(s)$$
$$y(s) = G_p\, G_f C(s) + G_d\, d(s)$$
$$y(s) = G_p\, G_f G_c E(s) + G_d\, d(s)$$
$$y(s) = G_p\, G_f G_c (y_{sp}(s) - y_m(s)) + G_d\, d(s)$$
$$y(s) = G_p\, G_f G_c (y_{sp}(s) - G_m y(s)) + G_d\, d(s)$$
$$y(s) = G_p\, G_f G_c y_{sp}(s) - G_p\, G_f G_c G_m y(s) + G_d\, d(s)$$
$$(1 + G_c\, G_f G_p G_m) y(s) = G_c\, G_f G_p\, y_{sp}(s) + G_d\, d(s)$$

$$\boxed{y(s) = \frac{G_c G_f G_p}{1 + G_c G_f G_p G_m} y_{SP}(s) + \frac{G_d}{1 + G_c G_f G_p G_m} d(s)}$$

Let $\mathbf{G = G_C\, G_f\, G_P}$

Let $\mathbf{G = G_C\, G_f\, G_P}$

$$\therefore\ y(s) = \frac{G}{1 + GG_m} y_{SP}(s) + \frac{G_d}{1 + GG_m} d(s)$$

$$\frac{G}{1 + GG_m} = G_{SP} \qquad \frac{G_d}{1 + GG_m} = G_{load}$$

Types of control proplems:

1) **Servo systems**: The distubance does not change (i.e. *0=(s)d*) while the set point undergoes change. The feedback controller act in such away as to keep y close to the changing $_{sp}y$. The T.F. of closed loop system of this type is

$$\bar{y}(s) = \frac{G_d}{1 + G_c G_f G_p G_m} \bar{d}(s)$$

$$\bar{y}(s) = G_{load}\, \bar{d}(s)$$

:

3) **Regulated systems**: In these systems the set point (desired value) is constant (*0=(s)y_{sp}*) and the change occurring in the load. The T.F. of closed loop control system of this type is:

$$\bar{y}(s)=\frac{G_d}{1+G_cG_fG_pG_m}\bar{d}(s)$$
$$\bar{y}(s)=G_{load}\,\bar{d}(s)$$

The feedback controller tries to eliminate the impact of the load change ***d*** to keep ***y*** at the desired setpoint.

Effect of controllers on the response of a controlled process: (1) Effect of Propertional Control The general T.F of the closed loop controller is:

$$\bar{y}(s)=\frac{G_cG_fG_p}{1+G_cG_fG_pG_m}\bar{y}_{SP}(s)+\frac{G_d}{1+G_cG_fG_pG_m}\bar{d}(s)$$

Consider $G_m=1$, $G_f=1$

Also for propertional controller $G_C=K_C$

And eqn. (*) becomes

$$\bar{y}(s)=\frac{K_cG_p}{1+K_cG_p}\bar{y}_{SP}(s)+\frac{G_d}{1+K_cG_p}\bar{d}(s)$$

For a first order systems

$$\tau_p\frac{dy}{dt}+y=K_pm+K_dd$$

Which gives

$$\bar{y}(s)=\frac{K_p}{\tau_ps+1}\bar{m}(s)+\frac{K_d}{\tau_ps+1}\bar{d}(s)$$

Thus for the uncontrolled system we have time constant= τ_p

Static gains: K_p for manipolation and K_d for load

put $G_p=\frac{K_p}{\tau_ps+1}$ and $G_d=\frac{K_d}{\tau_ps+1}$

Then by substitution in eqn. (**) and take the closed loop reponse as

$$\bar{y}(s)=\frac{K_c\frac{K_p}{\tau_ps+1}}{1+K_c\frac{K_p}{\tau_ps+1}}\bar{y}_{SP}(s)+\frac{\frac{K_d}{\tau_ps+1}}{1+K_c\frac{K_p}{\tau_ps+1}}\bar{d}(s)$$

$$\bar{y}(s)=\left[\frac{K_PK_C}{\tau_ps+1+K_pK_C}\bar{y}_{SP}(s)+\frac{K_d}{\tau_ps+1+K_pK_C}\bar{d}(s)\right]\times\frac{1+K_pK_C}{1+K_pK_C}$$

$$\bar{y}(s)=\frac{\frac{K_PK_C}{1+K_pK_C}}{\frac{\tau_ps}{1+K_pK_C}+\frac{1+K_pK_C}{1+K_pK_C}}\bar{y}_{SP}(s)+\frac{\frac{K_d}{1+K_pK_C}}{\frac{\tau_ps}{1+K_pK_C}+\frac{1+K_pK_C}{1+K_pK_C}}\bar{d}(s)$$

Rearrange the last eqn.

$$\bar{y}(s) = \frac{\bar{K}_P}{\bar{\tau}_p s + 1}\bar{y}_{SP}(s) + \frac{\bar{K}_d}{\bar{\tau}_p s + 1}\bar{d}(s)$$

Where

$$\bar{\tau}_p = \frac{\tau_P}{1 + K_p K_C} \quad \text{Closed loop time constant}$$

$$\bar{K}_p = \frac{K_p K_C}{1 + K_p K_C} \quad \text{Closed loop ststic gain}$$

$$\bar{K}_d = \frac{K_d}{1 + K_p K_C} \quad \text{Closed loop ststic gain}$$

The close-loop response has the following characteristics:-

1- It remains first order with respect to load and set point change

2- The time constant has been reduced ($\bar{\tau}_p < \tau_p$) which mean that the closed-loop response has become faster than the open loop response, to change in set point or load.

3- The static gain have been decreased.

Disadvantage of Proportional control Consider a servo problem with a unit step in the set point

$$\bar{y} = \frac{\bar{K}_p}{\tau_p s + 1} \cdot \frac{1}{s}$$

$$\bar{y}(t) = \bar{K}_p(1 - e^{-t/\tau_p})$$

$$\therefore \bar{y}(\infty) = \bar{K}_p$$

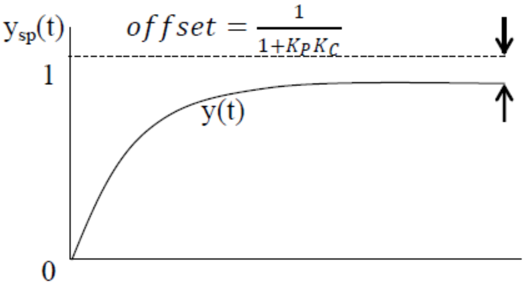
The ultimate response of $t \to \infty$ never reaches the desired new setpoint. There is always a discrepancy called ***offset*** which is equal to:

Offset = New set point - Ultimate value

$$= 1 - \bar{K}_p = 1 - \frac{K_p K_C}{1 + K_p K_C}$$

$$\therefore offset = \frac{1}{1 + K_p K_C}$$

Offset decreases as K_C becomes larger an thoretically $offset \to 0$ when $K_C \to \infty$



2- Effect of Integral Control

Consider a servo problem, $\bar{d}(s) = 0$

$$\bar{y}(s) = \frac{G_c G_f G_p}{1 + G_c G_f G_p G_m} \bar{y}_{SP}(s) \qquad (*)$$

Consider $G_m = G_f = 1$

For the 1st order process $G_p = \dfrac{k_p}{\tau_p s + 1}$

For a simple integral control

$$G_c = K_c \frac{1}{\tau_I s}$$

Sub in eqn. (*)

$$\bar{y}(s) = \frac{\dfrac{K_P}{\tau_p s + 1} \cdot \dfrac{K_C}{\tau_I s}}{1 + \dfrac{K_P}{\tau_p s + 1} \cdot \dfrac{K_C}{\tau_I s}} \bar{y}_{SP}(s) = \frac{K_P K_C}{(\tau_p s + 1)(\tau_I s) + K_P K_C} \bar{y}_{SP}(s)$$

$$\bar{y}(s) = \frac{\dfrac{K_P K_C}{K_P K_C}}{\dfrac{\tau_p \tau_I s^2}{K_P K_C} + \dfrac{\tau_I s}{K_P K_C} + \dfrac{K_P K_C}{K_P K_C}} \bar{y}_{SP}(s)$$

$$\bar{y}(s) = \frac{1}{\tau^2 s^2 + 2\psi \tau s + 1} \bar{y}_{SP}(s)$$

Where

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_p K_c}} \qquad \psi = \frac{1}{2} \sqrt{\frac{\tau_I}{\tau_p K_p K_c}}$$

Eqn. (**) indicates an important effect of the integral control action:-

1- It increases the order of the dynamic for the closed-loop reponse.
Thus for a first-order uncontrolled process, the response of the closed-loop becomes second order.
2- Increase K_c decreases ψ ∴more oscillatory
4- To examine the effect of integral on s.s error

$$\bar{y}(s) = \frac{1}{\tau^2 s^2 + 2\psi\tau s + 1}\bar{y}_{SP}(s)$$

If $\bar{y}_{SP}(s) = \frac{1}{s}$

The ultimate value= AK=1*1=1

∴offset= New setpoint-ultimate value

=1-1=0

It indicate that the integral control eliminates any offset

3- Effect of Derivative Control Action

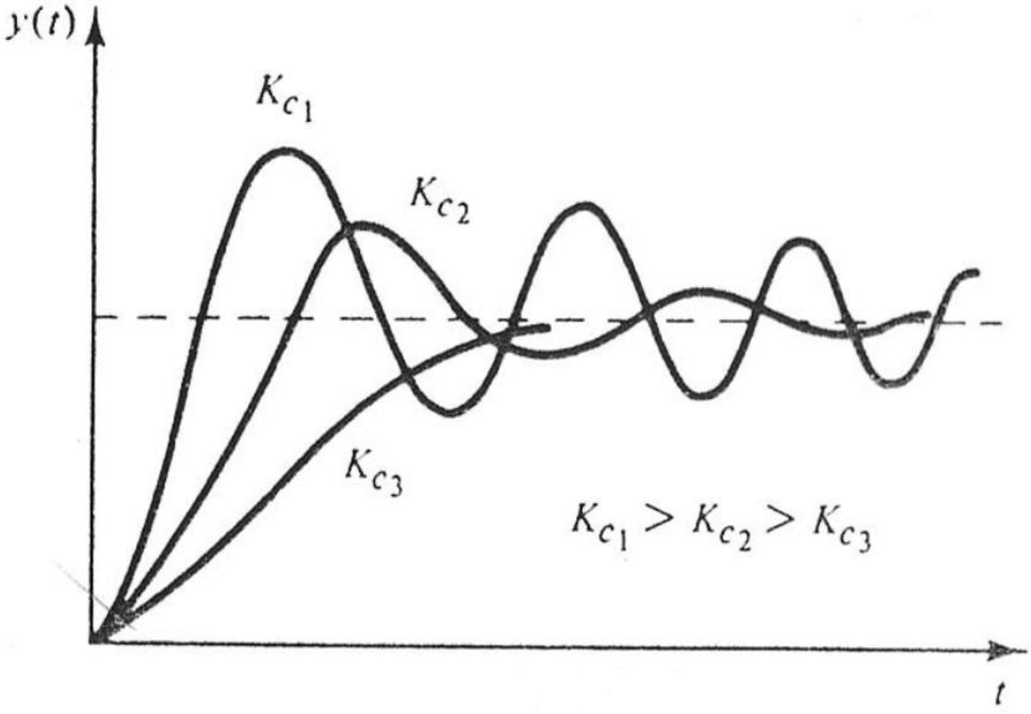
For derivative control

$$G_c = K_c\tau_D s$$

$$\bar{y}(s) = \frac{\frac{K_P}{\tau_p s + 1}\cdot K_c\tau_D s}{1 + \frac{K_P}{\tau_p s + 1}\cdot K_c\tau_D s}\bar{y}_{SP}(s) = \frac{K_P K_c \tau_D s}{\tau_p s + 1 + K_P K_c \tau_D s}\bar{y}_{SP}(s)$$

$$\bar{y}(s) = \frac{K_P K_c \tau_D s}{(\tau_p + K_p K_c \tau_D)s + 1}\bar{y}_{SP}(s)$$

1- The derivative control does not change the order of the reponse.
2- The effective time constant of the closed-loop response $(\tau_p + K_p K_c \tau_D) > \tau_p$ This means that the response of the controlled process is slower than that of the original first-order process and as K_c increase the response become slower.
Effect of Composite Control Action 1- Effect of PI control Combination of propertional and integral control modes lead to the follwing effects on the response of closed-loop system.
1- The order of the response inceases (effect of I mode).
2- The offset is eliminated (effect of I mode).
3- As K_c increses, the response becomes fater (effect of P and I modes) and more oscillatory to set point changes [ovesrshoot and decay ratio increase (effet of I mode)].
Large value of K_c create a very sensitive response and may lead to instability.
4- As τ_I decreases, for constant K_c, the reponse become faster but more oscillatory with higher overshoot and decay ratio (effect of I mode).
2- Effect of PID control To increase the speed of the closed loop response, increase the value of the controller gain K_c. But increasing enough K_c in order to have acceptable speed, the response become more oscillatory and may lead to unstability. The introduction of the derivative mode brings a stability effect to the system. Thus to achive
1- Acceptable response speed by selecting an appropriate value for the gain K_c.
2- While maintaining moderate overshoot and decay ratios.
3-



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Example: Regular loop with the following elements

$G_P(s) = \frac{3}{10s+1}$ (process)

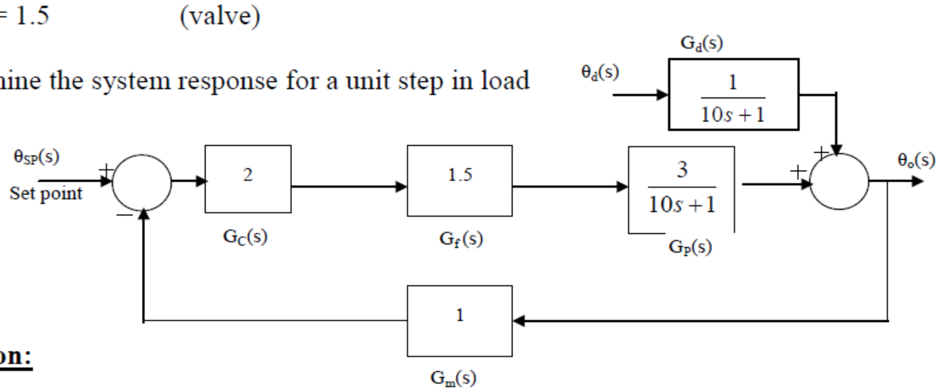
$G_d(s) = \frac{1}{10s+1}$ (Load)

$G_m(s) = 1$ (measuring device) if not given take 1

$G_C(s) = 2$ (controller)

$G_f(s) = 1.5$ (valve)

Determine the system response for a unit step in load



Solution:

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$\theta_o(s)$ $G_d(s)$

Regulator loop: $\dfrac{\theta_o(s)}{\theta_d(s)} = \dfrac{G_d(s)}{1+G_m(s)G(s)}$

$G(s) = G_C(s)\ G_f(s)\ G_P(s) = 2\times 1.5\times \dfrac{3}{10s+1} = \dfrac{9}{10s+1}$

$$\frac{\theta_o(s)}{\theta_d(s)} = \frac{\frac{1}{10s+1}}{1+\frac{9}{10s+1}} = \frac{1}{10s+10}$$

$\theta_d(s) = \dfrac{1}{s}$

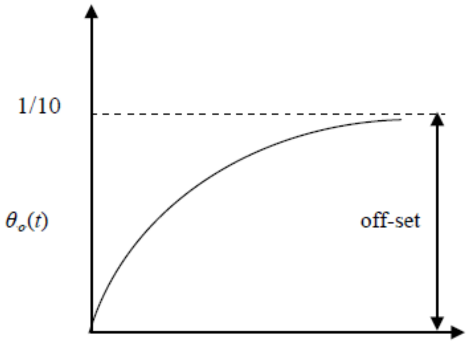
$$\theta_o(s) = \frac{\frac{1}{10}}{s(s+1)}$$

$\theta_o(t) = 0.1(1-e^{-t})$

At $t=0$, $\theta_o(t)=0$

At $t=\infty$, $\theta_o(\infty)=0.1$

Or



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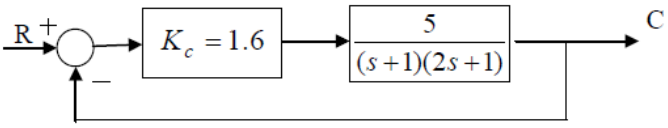
$$\theta_o(\infty) = \lim_{s\to 0} s\,\theta_o(s) = \lim_{s\to 0} s\frac{\frac{1}{10}}{s(s+1)} = \frac{1}{10}$$

Offset= New s.s value-Ultimate value= 0 - 0.1= -0.1

<u>Example</u>: the set point of the control system shown in the figure is gives a step change of a 0.1 unit. Determine

1- The maximum value of C.
2- The offset.
3- The period of oscillation.



$$\frac{C(s)}{R(s)} = \frac{G_cG_p}{1+G_cG_p}$$

$$\frac{C(s)}{R(s)} = \frac{1.6\times\frac{5}{(s+1)(2s+1)}}{1+1.6\times\frac{5}{(s+1)(2s+1)}} = \frac{8}{2s^2+3s+1+8}$$

$$\frac{C(s)}{R(s)} = \frac{8}{2s^2+3s+9} = \frac{\frac{8}{9}}{\frac{2}{9}s^2+\frac{1}{3}s+1} = \frac{0.8889}{0.222s^2+0.333s+1}$$

$\tau^2 = 0.222 \Rightarrow \tau = 0.471$

$2\psi\tau = 0.3333 \Rightarrow \psi = 0.3538$ (Underdamped)

Ultimate Value=A*K=0.1*0.8889=0.08889

$$Overshoot = \exp(\frac{-\psi\pi}{\sqrt{1-\psi^2}}) = \exp(\frac{-3.1418\times 0.3538}{\sqrt{1-(0.3538)^2}}) = 0.3047$$

1) The maximum value =Ultimate value*(1+*Overshoot*)
   = 0.08889*(1.3047)=0.1160

   To find the time required to reach maximum value apply K, A , C_{max}, ψ and τ in the equation.

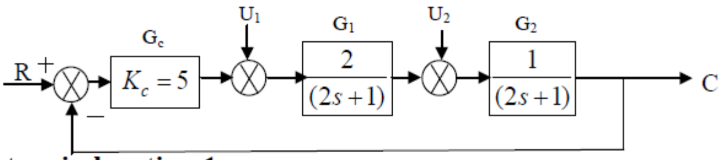
$$Y(t) = kA[1 - e^{(-\psi/\tau)t}(\cos wt + \frac{\psi}{\sqrt{1-\psi^2}}\sin wt]$$

$$w = \frac{\sqrt{1-\psi^2}}{\tau}$$

2) The offset=New set point- Ultimate value
   =0.1-0.088889=0.01111

3) Period of oscillation $= \frac{2\pi\tau}{\sqrt{1-\psi^2}} = \frac{2\pi\times 0.471}{\sqrt{1-(0.3538)^2}} = 3.1640$

Example: Consider the figure below, a unit step change in load enters at either location 1 or location 2.
What is the offset when the load enters at location 1 and when it enters at location 2



a-when the load enters in location 1

$$U_1(s) = \frac{1}{s} \quad , \quad U_2(s) = 0$$

$$C(s) = \frac{G_1G_2}{1+G_cG_1G_2}U_1(s)$$

$$C(s) = \frac{\frac{2}{2s+1}\cdot\frac{1}{2s+1}}{1+\frac{2}{2s+1}\cdot\frac{1}{2s+1}\times 5}U_1(s) = \frac{2}{4s^2+4s+1+10}U_1(s)$$

$$= \frac{2}{4s^2+4s+11}U_1(s) = \frac{2/11}{\frac{4}{11}s^2+\frac{4}{11}s+1}U_1(s)$$

$$K = \frac{2}{11} = 0.1818$$

$$\tau = \sqrt{\frac{4}{11}} = 0.6030$$

$$\tau = \sqrt{\frac{4}{11}} = 0.6030$$

$$2\psi\tau = \frac{4}{11} \Rightarrow \psi = \frac{4}{11} \times \frac{1}{2\tau} = 0.3015$$

Ultimate value=A.K=1*0.1818=0.1818

Offset=0 - 0.1818= -0.1818

<u>b-when the load enters in location 2</u>

$$C(s) = \frac{G_2}{1+G_cG_1G_2}U_2(s)$$

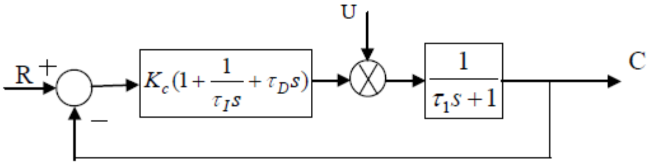
$$C(s) = \frac{\frac{1}{2s+1}}{1+\frac{2}{2s+1}\cdot\frac{1}{2s+1}\times 5}U_2(s) = \frac{2s+1}{4s^2+4s+1+10}U_2(s)$$

$$= \frac{2s+1}{4s^2+4s+11}U_2(s) = \frac{2s+1}{\frac{4}{11}s^2+\frac{4}{11}s+11}$$

$$C(\infty) = \lim_{s\to 0}\frac{2s+1}{\frac{4}{11}s^2+\frac{4}{11}s \ +11} = \frac{1}{11} = 0.091$$

Offset= 0 - 0.0.091= -0.091

<u>Example</u>: For the figure



For $\tau_D = \tau_I = 1 \quad and \tau_I = 2$

a- Calculate ψ when K_c=0.5 and K_c=2

b- Determine the effect for a unit-step change in load if K_c=2

$$\frac{C(s)}{R(s)} = \frac{G_cG_p}{1+G_cG_p}$$

$$\frac{C(s)}{R(s)} = \frac{K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}}{1+K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}} = \frac{K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}{1+K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}$$

$$= \frac{K_c(\frac{s+1+s^2}{s})\frac{1}{2s+1}}{1+K_c(\frac{s+1+s^2}{s})\frac{1}{2s+1}} = \frac{K_c(s+1+s^2)}{2s^2+s+K_c(s+1+s^2)} = \frac{K_c(s+1+s^2)}{(2+K_c)s^2+(1+K_c)s+K_c}$$

$$(s+1+s^2)$$

$$=\frac{(2+K_c)}{K_c}s^2+\frac{(1+K_c)}{K_c}s+1$$

a-1) K_c=0.5

$$\tau=\sqrt{\frac{2+K_c}{K_c}}=\sqrt{\frac{2+0.5}{0.5}}=2.2361$$

$$2\psi\tau=\frac{(1+K_c)}{K_c}=\frac{1+0.5}{0.5}=3\Rightarrow\psi=\frac{3}{2\tau}=\frac{3}{2\times2.2361}=0.6708$$

a-2) K_c=2

$$\tau=\sqrt{\frac{2+2}{2}}=1.4142$$

$$2\psi\tau=\frac{(1+K_c)}{K_c}=\frac{1+2}{2}=1.5\Rightarrow\psi=\frac{1.5}{2\tau}=\frac{1.5}{2\times1.41421}=0.5303$$

B) $C(s)=\frac{G_p}{1+G_pG_c}=U(s)$

$$C(s)=\frac{\frac{1}{\tau_1 s+1}}{1+K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}}U(s)=\frac{\frac{1}{2s+1}}{1+K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}U(s)$$

$$=\frac{s}{2s^2+s+K_c(s+1+s^2)}U(s)=\frac{s}{2s^2+s+2(s+1+s^2)}\times\frac{1}{s}$$

$$=\frac{1}{4s^2+3s+1}$$

$\tau=2$

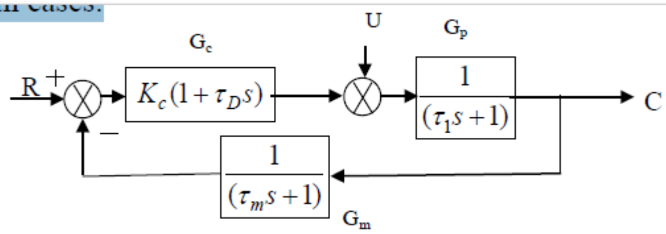
$$2\psi\tau=3\Rightarrow\psi=\frac{3}{4}=0.75$$

$$C(\infty)=\lim_{s\to0}s\frac{1}{4s^2+3s+1}=0$$

Offset=0-0=0

Example A PD controller is used in a control system having a first order process as shown. For Servo problem a-find expression for ψ and τ for the closed loop response. b-if τ_1=1 , τ_m=10 sec . Find K_c so that ψ=0.7 for two cases (1) τ_D=0 ,(2) τ_D=3 sec.
c- Calculate the offset in both cases.

c- Calculate the offset in both cases.



For the closed loop T.F.

$$C=\frac{G_cG_p}{1+G_cG_pG_m}R(s)=\frac{G_p}{1+G_cG_pG_m}U(s)$$

$$C=\frac{K_c(1+\tau_Ds)\cdot\dfrac{1}{\tau_1s+1}}{1+K_c(1+\tau_Ds)\cdot\dfrac{1}{\tau_1s+1}\cdot\dfrac{1}{\tau_ms+1}}R(s)$$

$$C=\frac{K_c(1+\tau_Ds)}{\tau_1s+1+K_c(1+\tau_Ds)\cdot\dfrac{1}{\tau_ms+1}}R(s)$$

$$C=\frac{K_c(1+\tau_Ds)}{\dfrac{\tau_1\tau_ms^2+(\tau_1+\tau_m)s+1+K_c+K_c\tau_Ds}{\tau_ms+1}}R(s)$$

$$C=\frac{K_c(1+\tau_Ds)(\tau_ms+1)}{\tau_1\tau_ms^2+(\tau_1+\tau_m+K_c\tau_D)s+(1+K_c)}R(s)$$

$$C=\frac{\dfrac{K_c(1+\tau_Ds)(\tau_ms+1)}{(1+K_c)}}{\dfrac{\tau_1\tau_ms^2}{(1+K_c)}+\dfrac{(\tau_1+\tau_m+K_c\tau_D)}{(1+K_c)}s+1}R(s)$$

$$\boxed{\tau = \sqrt{\frac{\tau_1 \tau_m}{1+K_c}}}$$

$$2\psi\tau = \frac{\tau_1 + \tau_m + K_c\tau_D}{1+K_c}$$

$$\psi = \frac{\tau_1 + \tau_m + K_c\tau_D}{2(1+K_c)} \sqrt{\frac{1+K_c}{\tau_1 \tau_m}}$$

$$\therefore \psi = \frac{\tau_1 + \tau_m + K_c\tau_D}{2\sqrt{(1+K_c)\tau_1\tau_m}}$$

b) $\therefore \psi = \dfrac{\tau_1 + \tau_m + K_c\tau_D}{2\sqrt{(1+K_c)}\sqrt{\tau_1\tau_m}}$ for ψ=0.7

1) τ_D=0

$$\therefore 0.7 = \frac{60+10+0}{2\sqrt{(1+K_c)}\sqrt{60\times 10}} = \frac{35}{\sqrt{600+600K_c}}$$

$$\sqrt{600+600K_c} = 50$$

$$600+600K_c = 2500$$

$$K_c = 3.166$$

2) τ_D=3 sec

$$\therefore 0.7 = \frac{60+10+3K_c}{2\sqrt{(1+K_c)}\sqrt{600}} = \frac{70+3K_c}{2\sqrt{(1+K_c)}\sqrt{600}}$$

$$70+3K_c = 34.292\sqrt{(1+K_c)}$$

$$2.04(1+0.042K_c) = \sqrt{(1+K_c)}$$

$$4.1616+0.355K_c+0.0075K_c^2 = (1+K_c)$$

$$0.0075K_c^2 - 0.0645K_c + 3.1616 = 0$$

$$\therefore K_c = 80.73 \quad or \quad K_c = 5.266$$

(c) The offset

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)$$

$$R = \frac{1}{s}$$

$$\text{Ultimate value} = \lim_{s\to 0} s \frac{K_c(1+\tau_D s)(\tau_m s+1) \big/ (1+K_c)}{\frac{\tau_1\tau_m s^2}{(1+K_c)} + \frac{(\tau_1+\tau_m+K_c\tau_D)}{(1+K_c)}s+1} \times \frac{1}{s} = \frac{K_c}{1+K_c} = \frac{3.166}{4.166} = 0.76$$

Offset= 1- 0.76= 0.24

Overall transfer function of a closed- loop control system

The transfer function of a block diagram is defined as the output divided by its input when represented in the Laplace domain with ze ro initial conditions. The transfer function G(s) of the block diagram shown in fig

$$\frac{Y(s)}{X(s)} = G(s)$$

Here the path of the signals X(s) and Y(s) is a forward path.

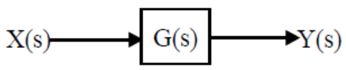


Fig. (1) Transfer function of a block diagram

Consider the block diagram of cascaded elements shown in Fig. (2a). Form the definition of a transfer function we have:

$$\frac{X_2(s)}{X_1(s)} = G_1(s)$$

$$\frac{X_3(s)}{X_2(s)} = G_2(s)$$

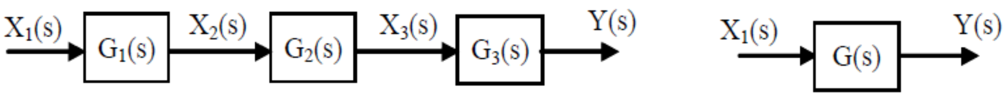
$$\frac{Y(s)}{X_3(s)} = G_3(s)$$

And substitution yields

$$Y(s) = G_3(s)X_3(s) = G_3(s)[G_2(s)X_2(s)] = G_3(s)G_2(s)G_1(s)X_1(s)$$

Which can be written as

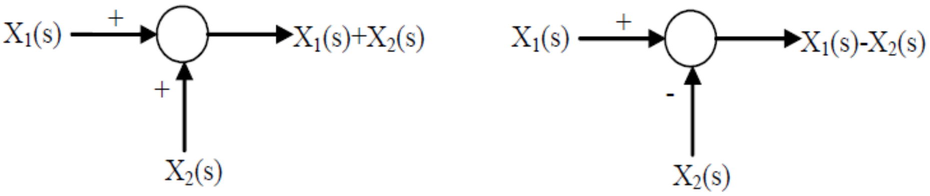
$$\frac{Y(s)}{X_1(s)} = G_3(s)G_2(s)G_1(s) = G(s)$$



(a) (b)

Fig. (2) Cascaded elements

The overall transfer function then is simply the product of individual transfer functions.
For applications where it is required to generate a signal which is the sum of two signals we define a summer or summing junction as shown in Fig. (3a). If the difference is required, then we define a subtractor as shown in Fig. (3b). Subtractors are often called error detecting devices since the output signal is the difference between two signals of which one is usually a reference signal.



(a) (b)

Summer Subtractor (Error detecting device)

Fig. (3) Addition or subtraction of signals

The combination of block diagrams in parallel is shown in Fig. (4a). Form the definition of the transfer function we have

$$Y_1(s) = G_1(s)X(s)$$
$$Y_2(s) = G_2(s)X(s)$$
$$Y_3(s) = G_3(s)X(s)$$

And the summer adds these signals,

$$Y(s) = Y_1(s) + Y_2(s) + Y_3(s)$$

or

$$Y(s) = [G_1(s) + G_2(s) + G_3(s)]X(s)$$

The overall transfer function shown in Fig.(4b) is

$$\frac{Y(s)}{X(s)} = G(s)$$

where

$$G(s) = G_1(s) + G_2(s) + G_3(s)$$

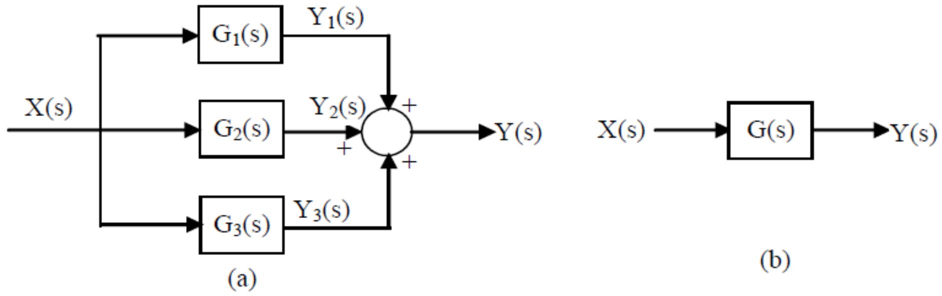


Fig. (4) Parallel combination of elements

In summary, we observe that for cascaded elements the overall transfer function is equal to the product of the transfer function of each element, whereas the overall transfer function for parallel elements is equal to the sum of the individual transfer function.

Example: Derive the overall transfer function for the control system shown in Fig.
(5).

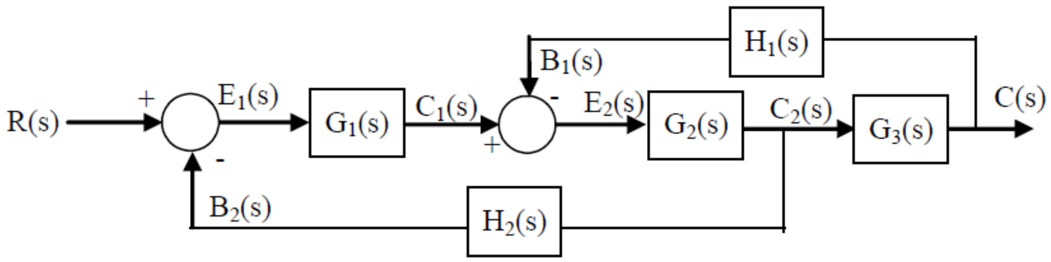


Fig.(5) Block diagram of a system with two feedback paths

Solution

$$E_1(s) = R(s) - B_2(s)$$
$$E_2(s) = C_1(s) - B_1(s)$$

$$C_1(s) = G_1(s)E_1(s)$$
$$C_2(s) = G_2(s)E_2(s)$$
$$C(s) = G_3(s)C_2(s)$$

$$B_1(s) = H_1(s)C(s)$$
$$B_2(s) = H_2(s)C_2(s)$$

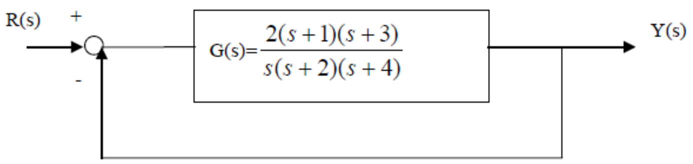
Substituting of the sub-transfer functions

$$C(s) = G_3(s)C_2(s)$$
$$C(s) = G_3(s)G_2(s)E_2(s)$$
$$C(s) = G_3(s)G_2(s)[C_1(s) - B_1(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)E_1(s) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)(R(s) - B_2(s)) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)R(s) - G_1(s)H_2(s)C_2(s) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)R(s) - G_1(s)H_2(s)\frac{C(s)}{G_3(s)} - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)G_1(s)R(s) - G_3(s)G_2(s)G_1(s)H_2(s)\frac{C(s)}{G_3(s)} - G_3(s)G_2(s)H_1(s)C(s)$$
$$[1 + G_2(s)G_1(s)H_2(s) + G_3(s)G_2(s)H_1(s)]C(s) = G_3(s)G_2(s)G_1(s)R(s)$$

Finally, the overall transfer function

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)G_2(s)H_2(s) + G_2(s)G_3(s)H_1(s)}$$

Example: A single-loop control system is shown in figure below. Determine closed-loop transfer function $\frac{Y(s)}{R(s)}$



Solution

Transfer function $\frac{Y(s)}{R(s)} = \frac{G}{1+GH}$

$$\frac{Y(s)}{R(s)} = \frac{\frac{2(s+1)(s+3)}{s(s+2)(s+4)}}{1 + \frac{2(s+1)(s+3)}{s(s+2)(s+4)} * 1} = \frac{\frac{2(s+1)(s+3)}{s(s+3)(s+4)}}{\frac{s(s+2)(s+4) + 2(s+1)(s+3)}{s(s+2)(s+4)}}$$

$$\frac{2(s+1)(s+3)}{s(s+2)(s+4) + 2(s+1)(s+3)} = \frac{2s^2 + 6s + 2s + 6}{s^3 + 4s^2 + 2s^2 + 8s + 2s^2 + 6s + 2s + 6}$$

$$\therefore \frac{Y(s)}{R(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

Block Diagram Reduction

When the block diagram representation gets complicated, it is advisable to reduce the diagram to a simpler and more manageable form prior to obtaining the overall transfer function. We shall consider only a few rules for block diagram reduction. We have already two rules, viz. Cascading and parallel connection. Consider the problem of moving the starting point of a signal shown in Fig. (6a) from behind to the front of G(s). since B(s)=R(s) and R(s)=C(s)/G(s), then B(s)=C(s)/G(s). therefore if the takeoff point is in front of G(s), then the signal must go through a transfer function 1/G(s) to yield B(s) as shown in Fig. (7b).

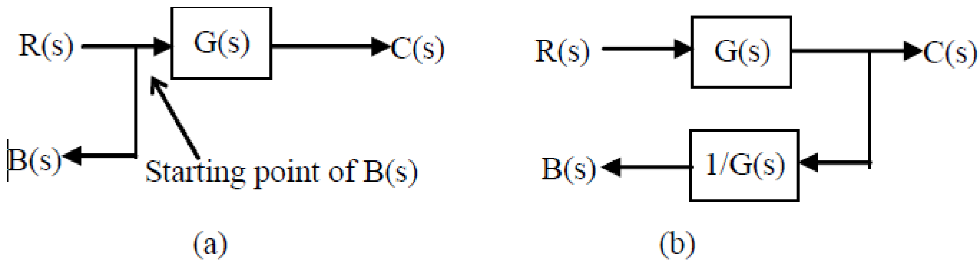


Fig.6 Moving the starting point of a signal

Consider the problem of moving the summing point of Fig. (7a). Since

$$E(s) = [M(s) + C(s)]G(s) = M(s)G(s) + C(s)G(s)$$

$$E(s) = M_1(s) + C_1(s)$$

where

$$M_1(s) = M(s)G(s); \qquad C_1(s) = C(s)G(s)$$

The generation of the signals $M_1(s)$ and $C_1(s)$ and adding them to yield $E(s)$ is shown in Fig. (7b). A table of the most common reduction rules is given in Table 1.

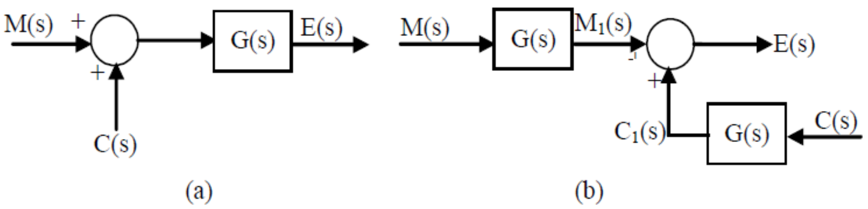
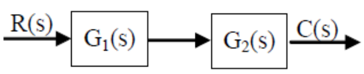
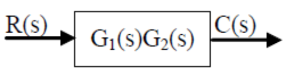
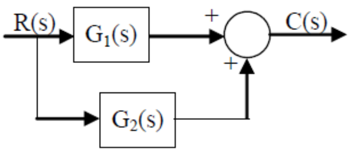
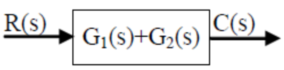
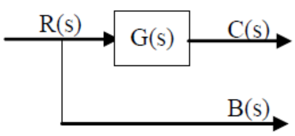
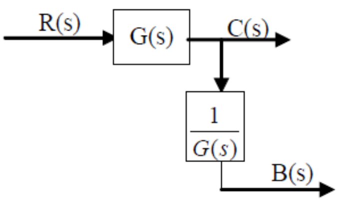
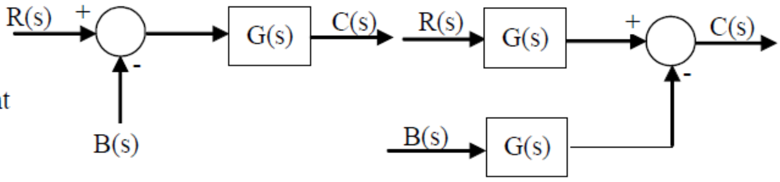


Fig.(7) Moving a summing junction

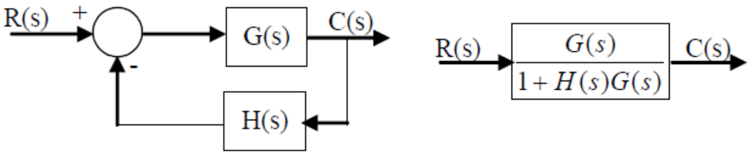
Table 1 Some rules for block diagram reduction

Rule	Original system	Reduced system
Cascaded elements		
Addition or subtraction		
Moving a starting point		

Moving a summing point



Closed loop system



Consider the transfer function of the system shown in Fig. (8a). The final transfer function is shown in Fig. (8d).

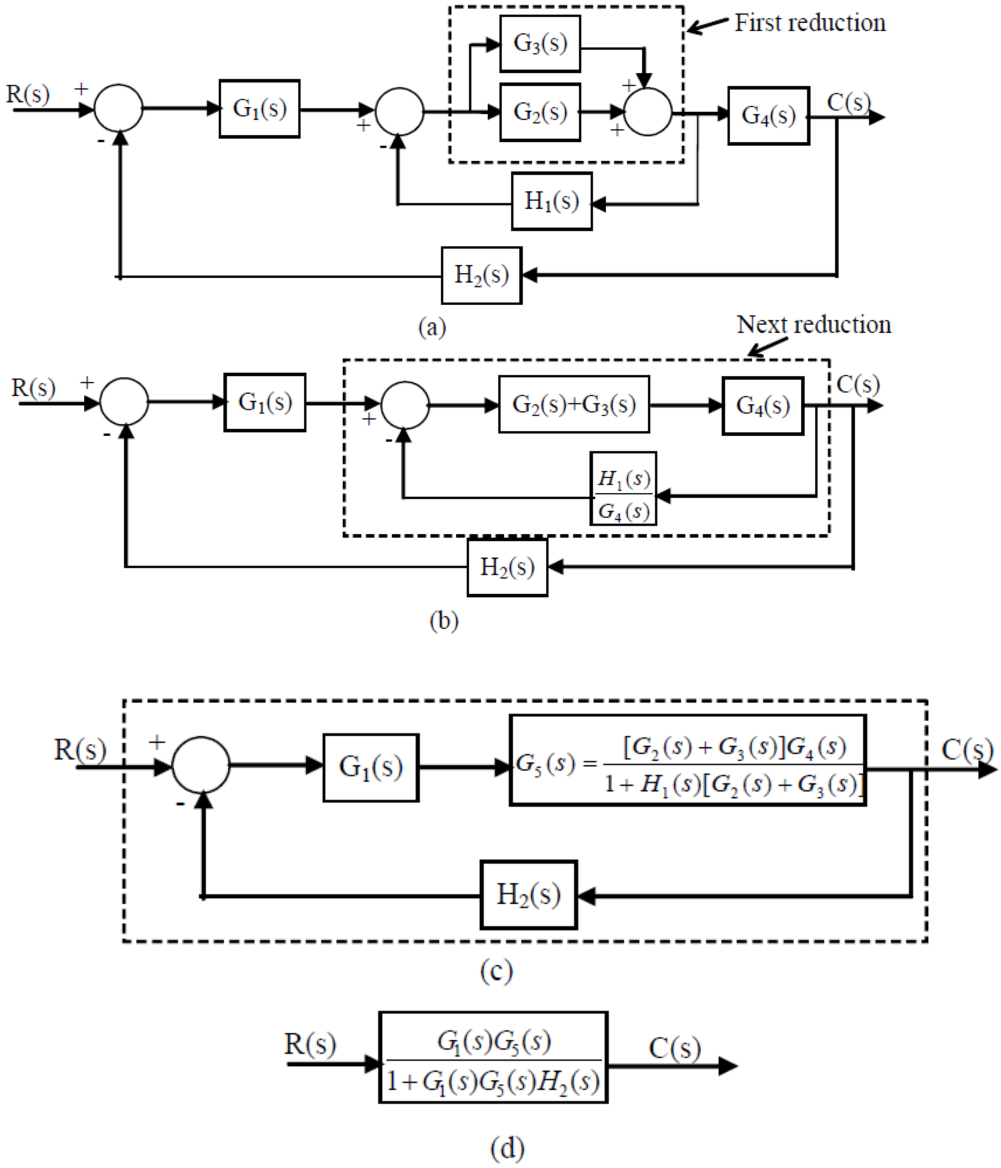
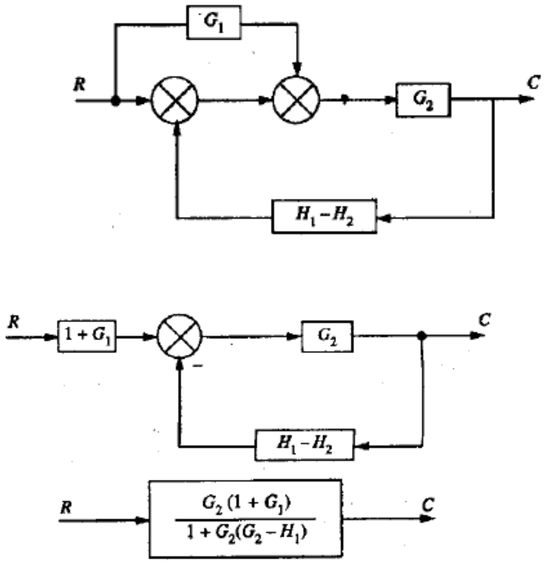
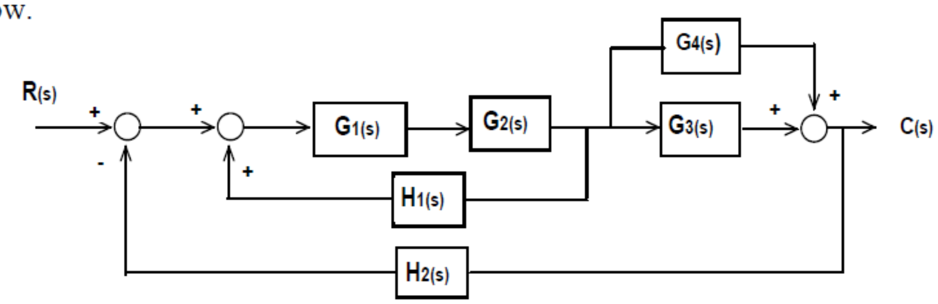


Fig.8 Obtaining transfer function by block diagram reduction

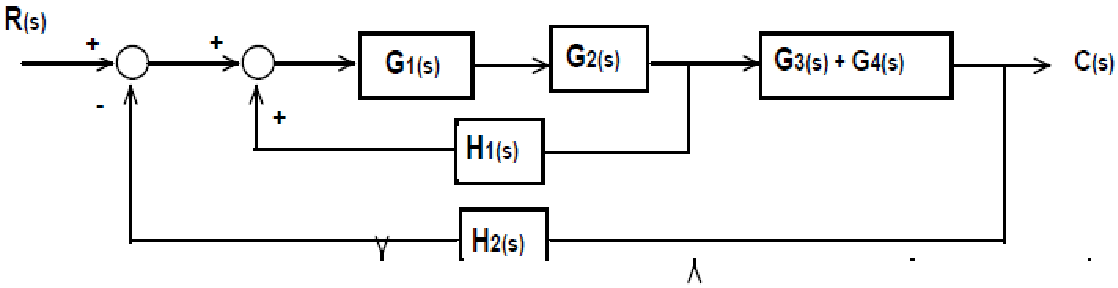
Example: Obtain the transfer function C/R of the block diagram shown in Figure below.

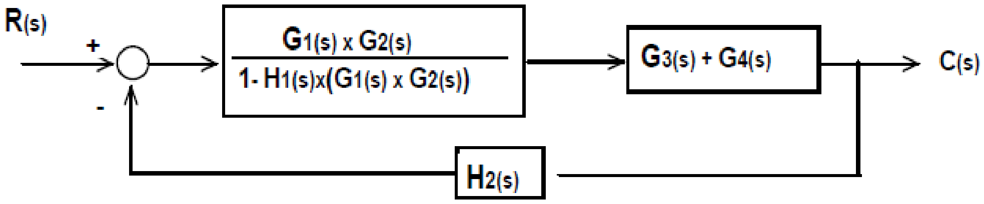


Example: Obtain the transfer function C/R of the block diagram shown in Figure below.



Solution:





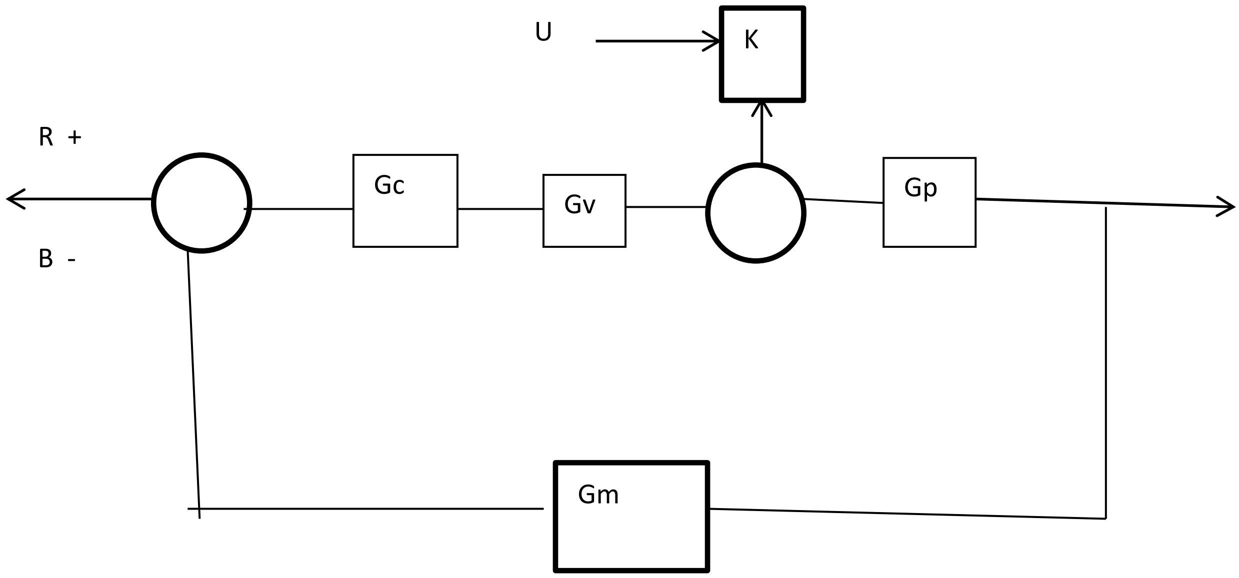
$$\frac{C(s)}{R(s)}=\frac{\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}{1+H_2(s)\left[\dfrac{G_1(s).G_2(s).\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}\right]}$$

$$=\frac{\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}{\dfrac{1-H_1(s).G_1(s).G_2(s)}{1-H_1(s)\cdot G_1(s)\cdot G_2(s)}+H(s)\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}$$

$$\frac{C(s)}{R(s)}=\frac{\left(G_1(s)\cdot G_2(s)\right)\times\left(G_1(s)+G_4(s)\right)}{\left(1-H_1(s)\cdot G_1(s)\cdot G_2(s)\right)+\left(H_2(s)\cdot G_1(s)\cdot G_2(s)\right)\times\left(G_3(s)+G_4(s)\right)}$$

Homework

Q1



Drive transfer function from the diagram below

a-for change in load u

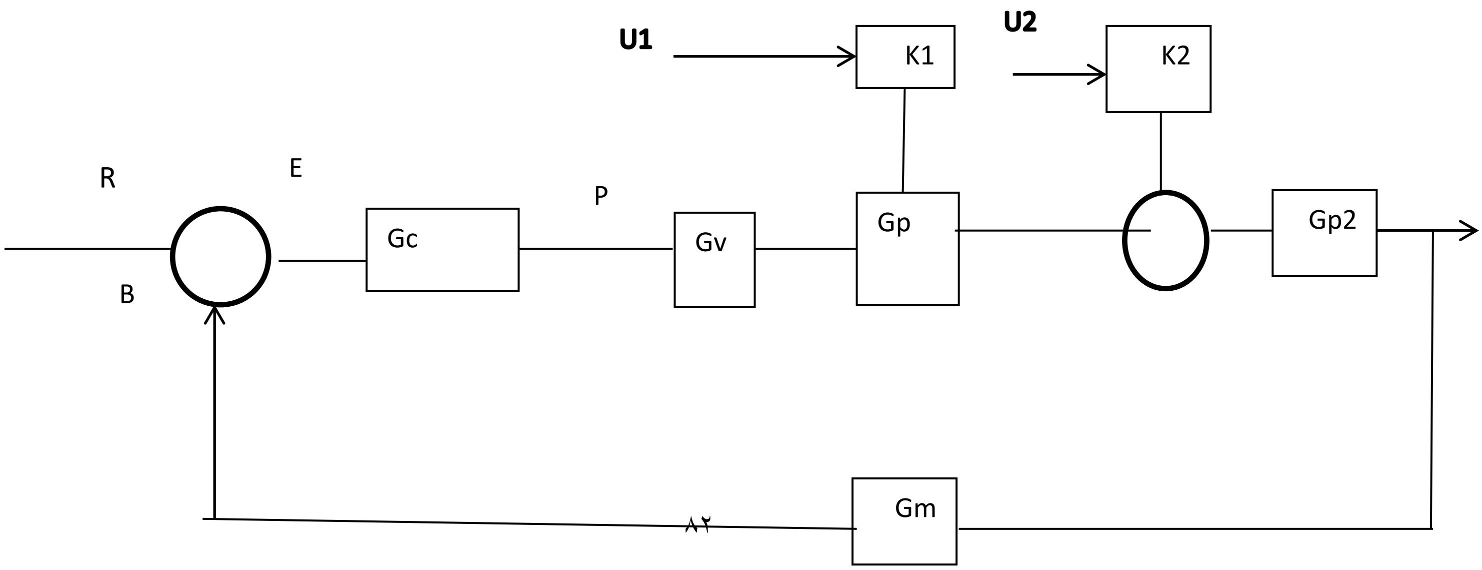
b- for change in set point

Q Drive transfer function from the diagram below

a-for change in load u1

b- for change in load u2

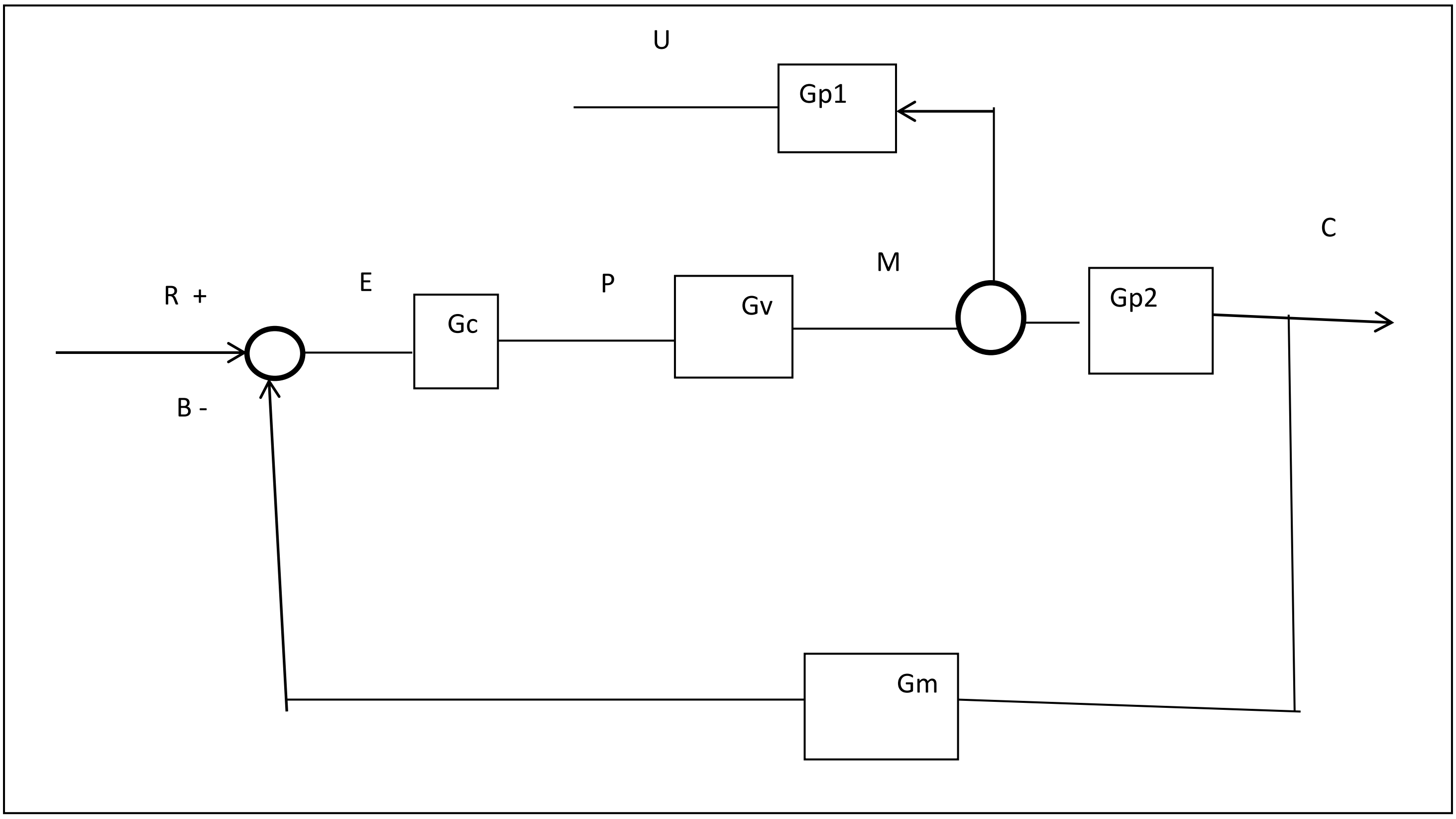
c- for change in set point



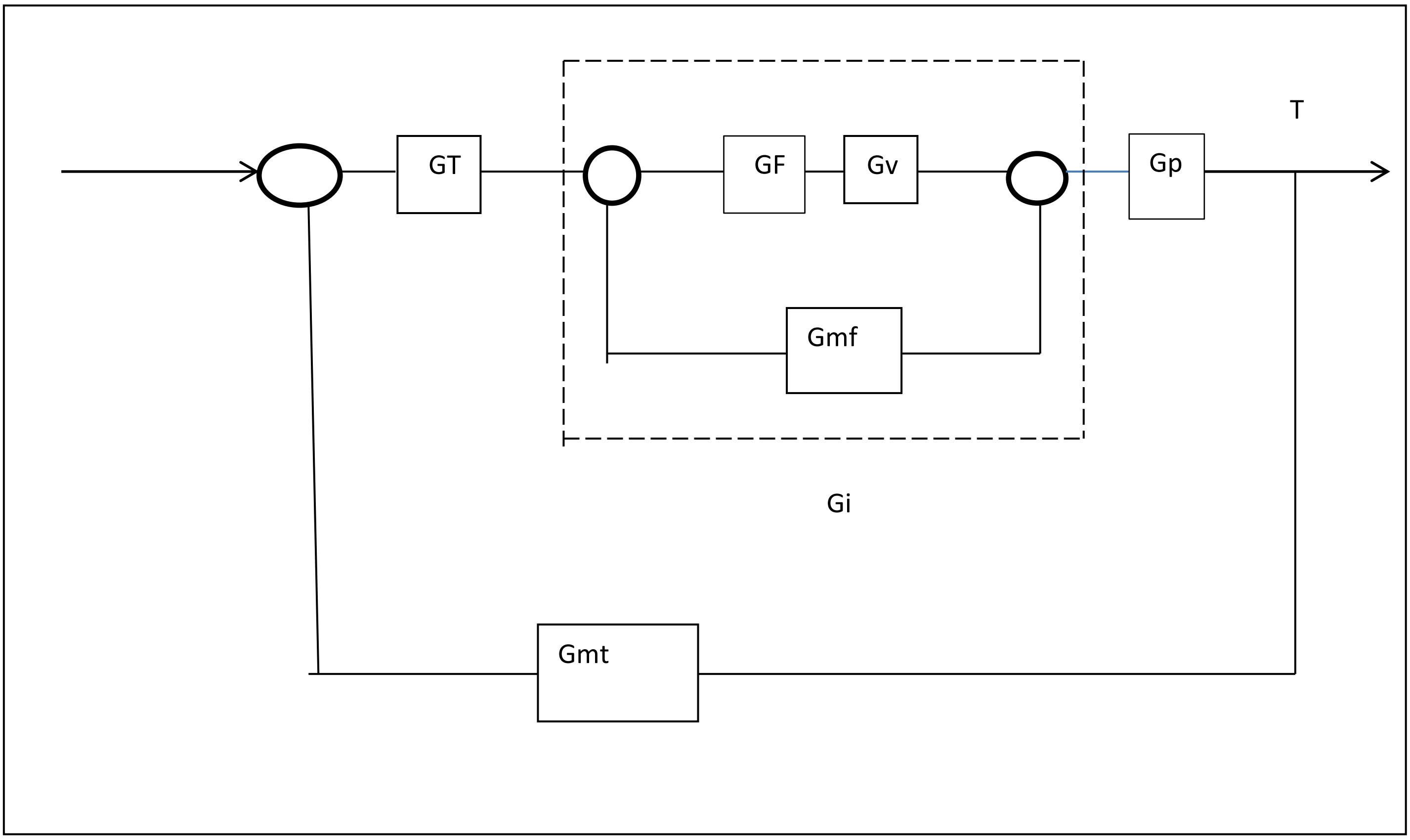
Q 3 Drive transfer function from the diagram below

a-for change in load u

b- for change in set point



Q4 state transfer function



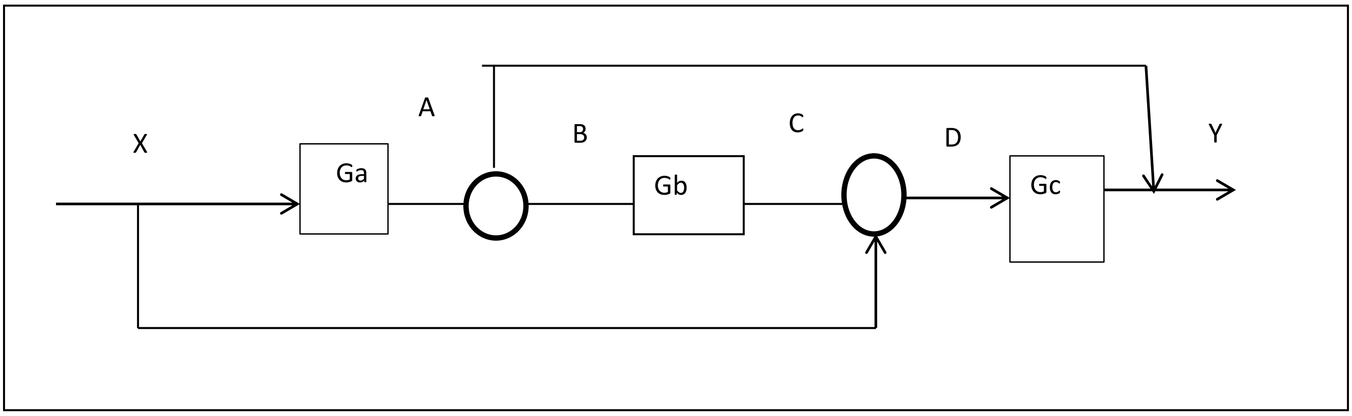
Q5 Sketch the block diagram and drive the feedback loop over all transfer function both with respect to set point change and process load change for each the following feed back control schemes

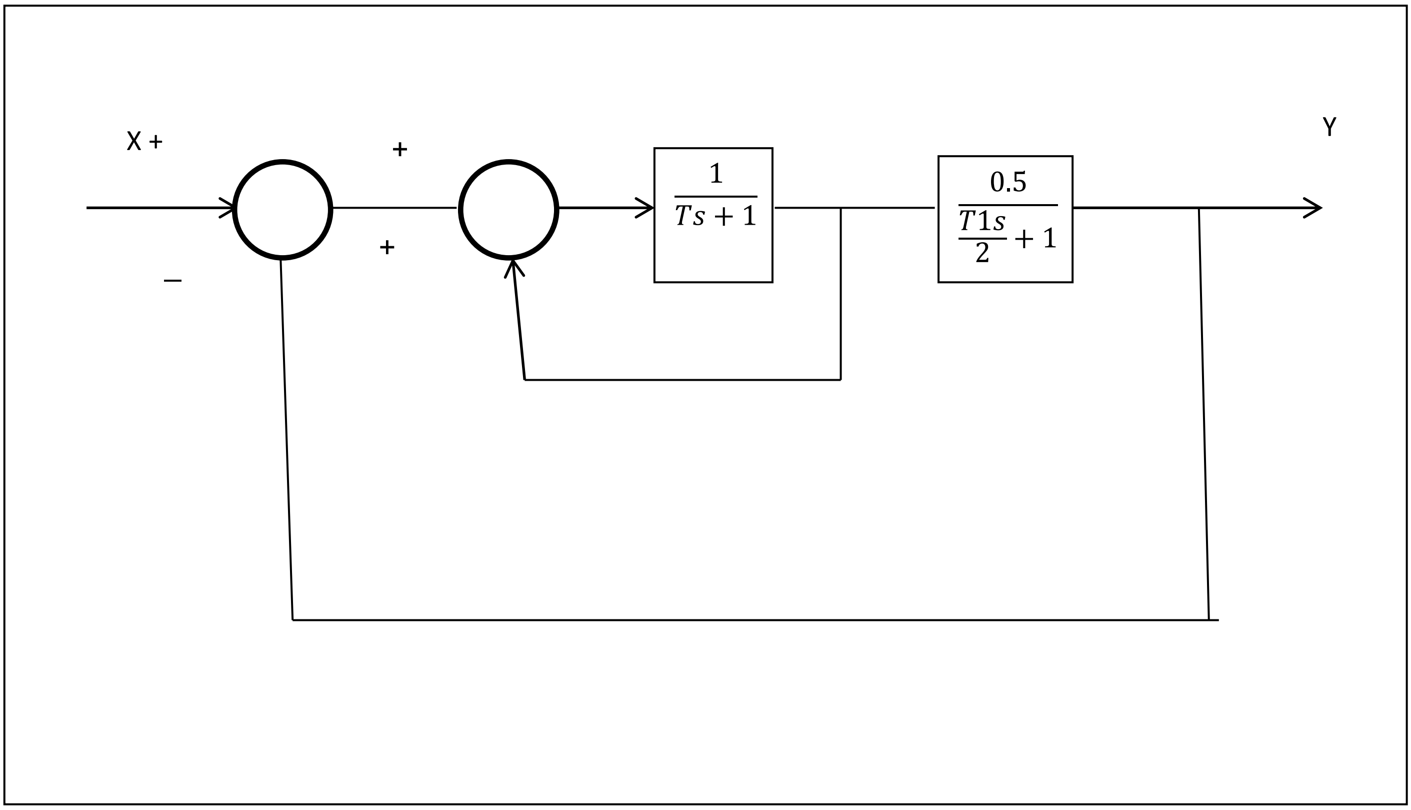
A-proptional-integral-derivative controller azero-order valve two first order process system in series with different gains and identical time constant and first order measuring element with transport lag giving a negative feed back signal

B-proptional –derivative controller whose outputs acts as asset point on an inner loop atrue second order process and zero-order measuring element giving a negative feed back signal

The inner loop containing proptional and first order valve and system contain three first order system with first order measuring elements giving negative sgin.

Q6 determine the transfer function





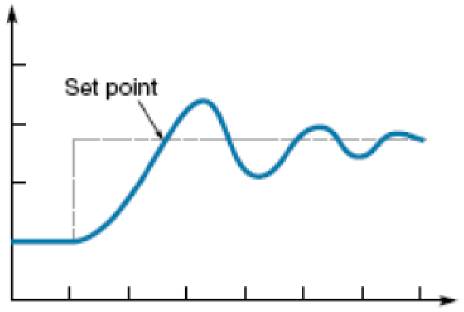
Repeat Q2 B-sketch the control loop block diagram for controlling t2 with control load MR the controller is proportional and the measuring element is first order with transport lag.

Repeat Q5 determine the parameter condition for the overall reaction system and the damping mode

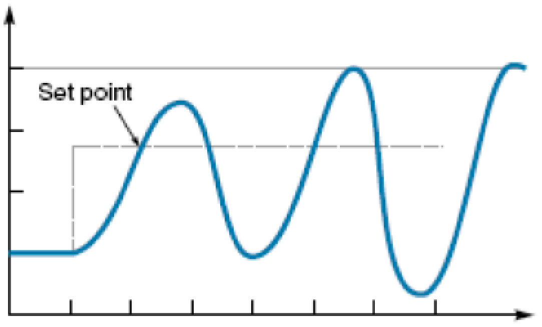
Repeat Q6 determine the parameter condition for the overall reaction system and the damping mode

Stability Analysis

A stable system is one where the controlled variable will always settle near the set point. An unstable system is one where, under some conditions, the controlled variable drifts away from the set point or breaks into oscillations that get larger and larger until the system saturates on each side.



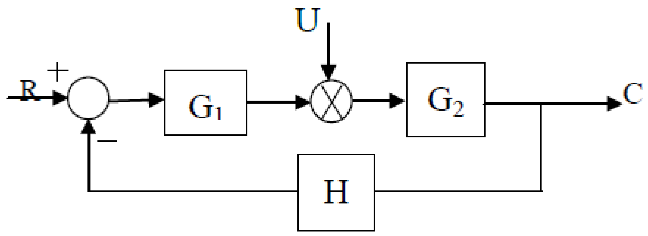
Stable system



unstable system

Methods of Stability Test

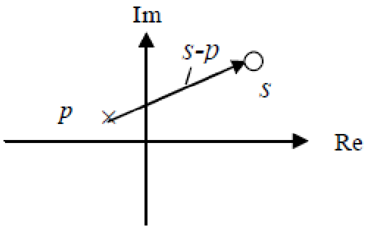
1-Determination the roots of equation



$$C = \frac{G_1G_2}{1+G_1G_2H}R(s) + \frac{G_2}{1+G_1G_2H}U(s)$$

$1+G_1G_2H = 0$ (Characterstic Equation)

$(s-r_1)(s-r_2)(s-r_3)\ldots\ldots\ldots = 0$



A linear control system is ***unstable*** if any roots of its characterstic equation are to the right of imaginary axis.

If this equation has some roots with positive real parts, then the system is unstable, or some roots equal to zero, the system is marginally stable (oscillatory), therefore it is unstable. Then for stability the roots of characteristic equation must have negative real parts.

<u>Example</u>:if

$G_1 = 10\frac{0.5s+1}{s}$ PI control

$G_2 = \frac{1}{2s+1}$ Stirred tank

$H = 1$ Mesuring element without lag

$1+G = 1+G_1G_2H = 0$

$1+\frac{10(0.5s+1)}{s(2s+1)} = 0$

$s(2s+1)+5s+10 = 0$

$2s^2+6s+10 = 0$

$s^2+3s+5 = 0$

$s = \frac{-3}{2} \mp \frac{\sqrt{9-20}}{2}$

$\therefore s_1 = \frac{-3}{2} + j\frac{\sqrt{11}}{2}$ and $s_2 = \frac{-3}{2} - j\frac{\sqrt{11}}{2}$

2-Routh's Method

a-Write the characterstic eqn. on the form of a polynomial shape:

$$a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots . a_n = 0 \qquad (*)$$

Where a_o is positive

It is necessary that a_o, a_1, a_2,…. a_{n-1}, a_n be positive. If any coeff. is negative, the system is ***unstable***.

If all of the coeff. are positive, the system may be stable or unstable. Then apply the next step.

b. Routh array:

Arrange the coeff. of eqn. (*) into the first two rows of the Routh array shown below.

Row				
1	a_o	a_2	a_4	a_6
2	a_1	a_3	a_5	a_7
3	A_1	A_2	A_3	
4	B_1	B_2	B_3	
n+1	C_1	C_2	C_3	

$$A_1 = \frac{a_1 a_2 - a_o a_3}{a_1} \quad , \quad A_2 = \frac{a_1 a_4 - a_o a_5}{a_1} \quad , \quad A_3 = \frac{a_1 a_6 - a_o a_7}{a_1}$$

$$B_1 = \frac{A_1 a_3 - a_1 A_2}{A_1} \quad , \quad B_2 = \frac{A_1 a_5 - a_1 A_3}{A_1}$$

$$C_1 = \frac{B_1 A_2 - A_1 B_2}{B_1} \quad , \quad C_2 = \frac{B_1 A_3 - A_1 B_3}{B_1}$$

Examine the elements of the first column of the array a_o, a_1, A_1, B_1,C_1……….W_1

a) If any of these elements is negative, we have at least one root on the right of the imaginary axis and the system is unstable.

b) The number of sign changes in the elements of the first column is equal to the number of root to the right of the imaginary axis.

∴The system is ***stable*** if all the elements in the first column of the array are positive

Example: Given the characteristic eqn.

$s^4 + 3s^3 + 5s^2 + 4s + 2 = 0$

Solution:

Row			
1	1	5	2
2	3	4	0
3	11/3	2	0
4	2.36	0	
5	2		

$$A_1 = \frac{3\times 5 - 4\times 11}{3} = \frac{11}{3}$$

$$A_2 = \frac{3\times 2 - 0}{3} = 2$$

$$B_1 = \frac{11/3\times 4 - 6}{11/3} = 2.36$$

$$C_1 = \frac{2.36\times 2}{2.36} = 2$$

$\therefore$ The system is **stable**

Example: Apply the Routh's stability criterion to the equation:

$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

Solution:

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	-6	0	
s^0	5		

The system is unstable.

Example: A system has a characteristic equation $s^3 + 9s^2 + 26s + 24 = 0$. Using the Routh criterion, show that the system is stable.

Solution

$q(s) = s^3 + 9s^2 + 26s + 24$

Using the Routh-Hurwitz criterion,

s^3	1	26
s^2	9	24
s^1	26	0
s^0	24	0

No sign change in 1^{st} column then the system is stable.

Example: Consider the feedback control system with the characteristic equation.

$$s^3 + 2s^2 + (2 + K_c)s + \frac{K_c}{\tau_I} = 0$$

Solution:

The corresponding Routh array can now be formed

Row			
1	1	$2 + K_c$	0
2	2	$\frac{K_c}{\tau_I}$	0
3	$\frac{2(2+K_c) - K_c/\tau_I}{2}$	0	0
4	K_c/τ_I	0	0

The elemnets of the first-column are positive except the third, which can be positive or negative depending on K_c and τ_I.

So state the stability

Put $\frac{2(2+K_c) - K_c/\tau_I}{2} > 0 \quad \Rightarrow 2(2+K_c) > \frac{K_c}{\tau_I}$

If K_c and τ_I satisfy the condition, then the system is atable

Critical stability

Put the third element=0

i.e $2(2+K_c)=\dfrac{K_c}{\tau_I}$

For $\tau_I=0.1$

$2(2+K_c)=10K_c \Rightarrow 4=8K_c$

$K_c=0.5$

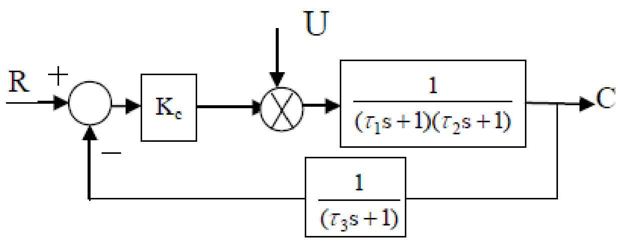
---

1) if $K_c<0.5$, the system is stable (all of the elements in the 1st column is +ve)

2) if $K_c>0.5$, the third element of the 1st column is negative. We have two sign change in the elements of the first column.

   $\therefore$ we have two roots to the right of imaginary axis.

Example:



If $\tau_1=1,\ \tau_2=\dfrac{1}{2},\ \tau_3=\dfrac{1}{3}$

Determine K_c for a stable system

Solution:

The char. Eqn.

$$1+K_c\frac{1}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}=0$$

$$(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)+K_c=0$$

$$(\frac{1}{2}s^2+\frac{3}{2}s+1)(\frac{1}{3}s+1)+K_c=0$$

$$\frac{s^3}{6}+\frac{s^2}{2}+\frac{s}{3}+\frac{s^2}{2}+\frac{3s}{2}+1+K_c=0$$

$$\frac{1}{6}s^3+s^2+\frac{11}{6}s+1+K_c=0$$

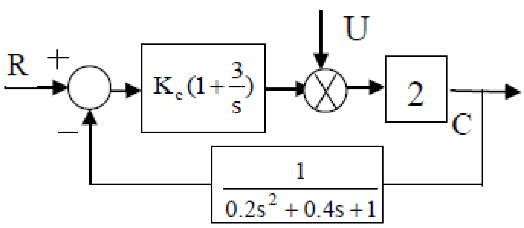
Row		
1	1/6	11/6
2	1	$1+K_c$
3	$\frac{10-K_c}{6}$	0
4	$1+K_c$	

Since $K_c>0$ ∴The system will be stable If $10-K_c>0$

$$K_c<10$$

Therfore K_c must within the range $0<K_c<10$

Example:



Study the stability for $K_c=2$

Solution:

$$1+K_c(1+\frac{3}{s})\times 2\times\frac{1}{0.2s^2+0.4s+1}=0$$

$$1+K_c(\frac{s+3}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$1+(\frac{sK_c+3K_c}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$0.2s^3+0.4s^2+s+2sK_c+6K_c=0$$

$$0.2s^3+0.4s^2+(1+2K_c)s+6K_c=0$$

Row		
1	0.2	$1+2K_c$
2	0.4	$6K_c$
3	A_1	0
4	B_1	0

Row	For $K_c=2$	
1	0.2	5
2	0.4	12
3	$\frac{2-2.4}{0.4}$	0
4	1.2	0

$$A_1=\frac{0.4(1+2K_c)-(1.2K_c)}{0.4}=\frac{0.4+0.8K_c-1.2K_c}{0.4}=\frac{0.4-0.4K_c}{0.4}$$

$0.4-0.4K_c>0$

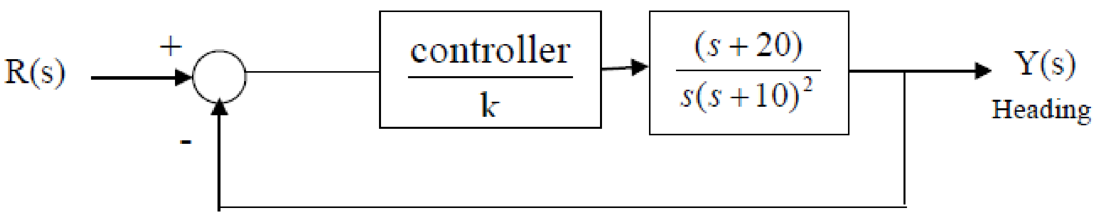
The system is stable for $K_c<1$

$B_1=6K_c \Rightarrow 6K_c>0$

And $K_c>0$

Therfore K_c must within the range $0<K_c<1$

Example: Designers have developed small, fast, vertical-take off fighter aircraft that are invisible to radar. This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown in figure. Determine the maximum gain of the system for stable operation.



Solution

$$G(s)=\frac{k(s+20)}{s(s+10)^2}=\frac{ks+20k}{s(s^2+20s+100)}=\frac{ks+20k}{s^3+20s^2+100s}$$

Characteristic equation,

$1+GH=0$

$$1+\frac{ks+20k}{s^3+20s^2+100s}*1=0$$

$s^3+20s^2+100s+ks+20k=0$

$s^3+20s^2+(100+k)s+20k=0$

The corresponding Routh array can now be formed

Row		
1	1	100+k
2	20	20k
3	a	0
4	b	0

$$a = \frac{20(100+k)-20k}{20} = \frac{20*100+20k-20k}{20} = 100$$

$$b = \frac{a*20k-0}{a} = 20\,k$$

The system is stable, no sign change in 1^{st} column,

b>0

20k >0

k>0

$\therefore$ Range of k is must be k>0

Frequency Response Analysis

It is how the output response (amplitude, phase shift) change with the frequency of the input sinusoidal. The input frequency is varied, and the output characteristics are computed or represented as a function of the frequency. Frequency response analysis provides useful insights into stability and performance characteristics of the control system. Figure below shows the hypothetical experiment that is conducted.

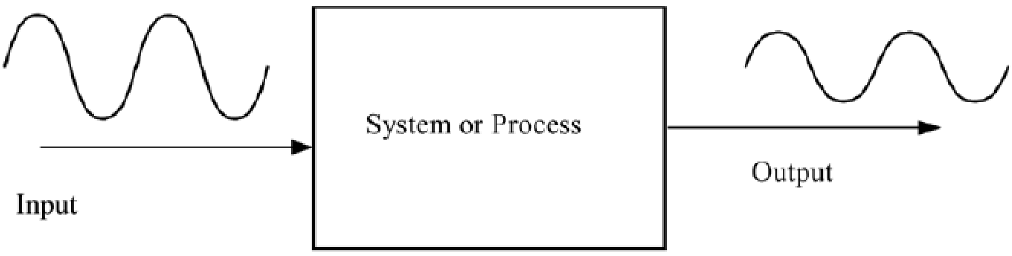


Figure: How frequency response is defined.

Response of a first-Order System to a Sinusoidal Input

Consider a simple first-order system with a transfer function

$$G(s) = \frac{\bar{y}(s)}{\bar{F}(s)} = \frac{K_p}{\tau_p s + 1} \qquad (1)$$

Let F(t) be a sinusoidal input with amplitude A and frequency ω;

$F(t) = A \sin(\omega t)$

Then

$$\bar{F}(s) = \frac{A\omega}{s^2 + \omega^2} \qquad (2)$$

Sub. $\bar{F}(s)$ from eq. (2) into eq. (1)

$$\bar{y}(s) = \frac{K_p}{\tau_p s + 1} \times \frac{A\omega}{s^2 + \omega^2} = \frac{K_p}{\tau_p s + 1} \times \frac{A\omega}{(s + j\omega)(s - j\omega)}$$

Expand into partial fraction and find

$$\bar{y}(s) = \frac{C_1}{s + 1/\tau_p} + \frac{C_2}{s + j\omega} + \frac{C_3}{s - j\omega}$$

Compute the constants C_1, C_2 and C_3 and find the inverse of laplace transform.

$$C_1 = \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \quad , \quad C_2 = \frac{-K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \quad , \quad C_3 = \frac{K_p A}{\tau_p^2 \omega^2 + 1}$$

$$\bar{y}(t) = \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} e^{-t/\tau_p} - \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1} \sin(\omega t)$$

As $t \to \infty$, then $e^{-t/\tau_p} \to 0$, and the first term disappears.

Thus, after a long time, the response of a first order system to a sinusoidal input is given by:

$$\bar{y}_{ss}(t) = -\frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1}\cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1}\sin(\omega t)$$

$$\bar{y}_{ss}(t) = \frac{K_p A}{\tau_p^2 \omega^2 + 1}[-\omega \tau_p \cos(\omega t) + \sin(\omega t)] \quad (3)$$

Use the follwing trigonometric identity.

$$p\cos\theta + q\sin\theta = r\sin(\theta + \phi)$$
$$r = \sqrt{p^2 + q^2} \qquad \phi = \tan^{-1}\frac{p}{q}$$

$$q = 1 \qquad p = -\omega \tau_p$$

$$r = \sqrt{(-\tau_p \omega)^2 + (1)^2} = \sqrt{\tau_p^2 \omega^2 + 1}$$

$$\phi = \tan^{-1}\frac{p}{q} = \tan^{-1}(\frac{-\omega \tau_p}{1}) = \tan^{-1}(-\omega \tau_p)$$

Then eq.(3) yield

$$\bar{y}_{ss}(t) = \frac{K_p A}{\tau_p^2 \omega^2 + 1}[(\sqrt{\tau_p^2 \omega^2 + 1})\sin(\omega t + \phi)]$$

$$\bar{y}_{ss}(t) = \frac{K_p A}{\sqrt{\tau_p^2 \omega^2 + 1}}\sin(\omega t + \phi) \quad (4)$$

$$\phi = \tan^{-1}(-\omega \tau_p) = \text{Phase lag} \quad (5)$$

From eq.(4) and eq. (5), we observe that:

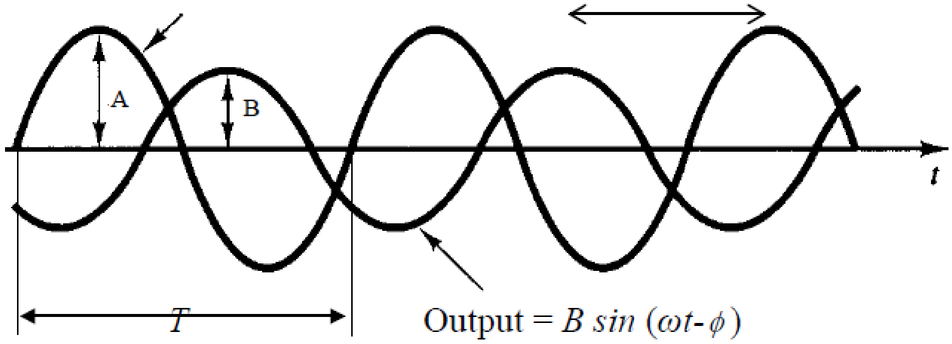
1) The ultimate response (also referred to as s.s.) of a first order system to a sin input is also a sinusoidal wave with the same frequency ω.
2) The ratio of the output amplitude to the input amplitude is called the " amplitude ratio" and is a function of the frequency:

$$AR = \text{amplitude ratio} = \frac{\dfrac{K_p A}{\sqrt{\tau_p^2 \omega^2 + 1}}}{A} = \frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}}$$

$$AR = \frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}} \quad (6)$$

3) The output wave lags behind (phase lag) the input wave by an angle φ>, which is a function of the frequency ω(see eq.(5)).

Input = *A sin (ωt)*φ



It is the most important methods for stability analysis and used for design purposes control system.

Suppose the input to the process is sinusoidal signal

Where: A is amplitude

ω is frequency (rad/sec) = $\frac{1}{T}$

T is period of one complete cycle (time)

Frequency Response of a Second Order System

For a second order system the transfer function is:

K

For a second order system the transfer function is:

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$

Put s=jω then

1) Amplitude Ratio

$$AR = \frac{K_p}{\sqrt{(1-\tau^2\omega^2)^2 + (2\psi\tau\omega)^2}}$$

2) Phase shift

$$\phi = \tan^{-1}\left(-\frac{2\psi\tau\omega}{1-\tau^2\omega^2}\right)$$

Which is aphase lag since $\phi < 0$

Frequency Response of a Pure Dead-Time Process

$$G(s) = e^{-t_d s}$$

Put s=jω

$$\therefore \quad G(j\omega) = e^{-jt_d\omega}$$

$$\boxed{AR = 1}$$

$$\boxed{\phi = -t_d \omega}$$

That is aphase lag since $\phi < 0$

Frequency Response of a Feedback Controllers

1- **Propertional controller:**

The transfer function is $G(s) = K_c$

$$\therefore \quad \boxed{AR = K_c}$$

$$\boxed{\phi = 0}$$

2- **PI controller:**

The transfer function is $G(s) = K_c(1 + \frac{1}{\tau_I s})$

$$\therefore \quad \boxed{AR = K_c \sqrt{1 + \frac{1}{(\omega \tau_I)^2}}}$$

$$\boxed{\phi = \tan^{-1}(\frac{-1}{\omega \tau_I})} \quad < 0$$

3- **PD controller:**

3- PD controller:

The transfer function is $G(s) = K_c(1+\tau_D\, s)$

$$AR = K_c\sqrt{1+\tau_D^2\omega^2}$$

$$\phi = \tan^{-1}(\tau_D\,\omega) \qquad > 0$$

The positive phase shift is called phase lead and implies that the controller output lead the input.

4-PID controller:

The transfer function is $G(s) = K_c(1+\dfrac{1}{\tau_I\, s}+\tau_D\, s)$

$$AR = K_c\sqrt{(\tau_D\omega-\frac{1}{\tau_I\omega})^2+1}$$

$$\phi = \tan^{-1}(\tau_D\omega-\frac{1}{\tau_I\omega})$$

ϕ is + or – ve depending on the values of τ_D, τ_I and ω

Bode Diagrams

The bode diagrams consist of a pair of plots showing: 1. How the logarithm of the amplitude ratio varies with frequency. 2. How the phase shift varies with frequency.

First Order system:

Amplitude ratio $AR = \dfrac{K_p}{\sqrt{1+\tau_p^2\omega^2}}$ (*)

Phase lag= $\phi = \tan^{-1} - \tau_p\,\omega$

$\log\dfrac{AR}{K_p} = -\dfrac{1}{2}\log(1+\tau_p^2\,\omega^2)$ (**)

The plot can be carried by considering its asymptotic behaviour as **$\omega\to 0$** and as **$\omega\to\infty$** . Then

1. As $\omega\to 0$, then $\tau_p\,\omega \to 0$ and from eq.(*)

$\log\dfrac{AR}{K_p}\to 0$ or $\dfrac{AR}{K_p}=1$.This is the low-frequency asymptote. It is a horizontal line passing through the point $\dfrac{AR}{K_p}=1$.

2. As $\omega\to\infty$, then $\tau_p\,\omega\to\infty$ and from eq.(**)

$\log\frac{AR}{K_p} = -\log\tau_p\,\omega$. This is the high frequency asymptote.

It is a line with slope -1 passing through the point $\frac{AR}{K_p} = 1$ for $\tau_p\,\omega = 1$.

3. At the corner $\tau_p\omega = 1 \rightarrow \omega = \omega_c$

$$\omega_{corner} = \omega_c = \frac{1}{\tau_p}$$

The frequency ω_c is known as the corner frequnecy (and $\frac{AR}{K_p} = \frac{1}{\sqrt{2}} = 0.707$)

The phase lag plot

$as\ \omega \rightarrow 0 \quad , \quad \phi \rightarrow 0$

$as\ \omega \rightarrow \frac{1}{\tau_p} \quad , \quad \phi \rightarrow \tan^{-1}(-1) = -45^o$

$as\ \omega \rightarrow \infty \quad , \quad \phi \rightarrow \tan^{-1}(-\infty) = -90^o$

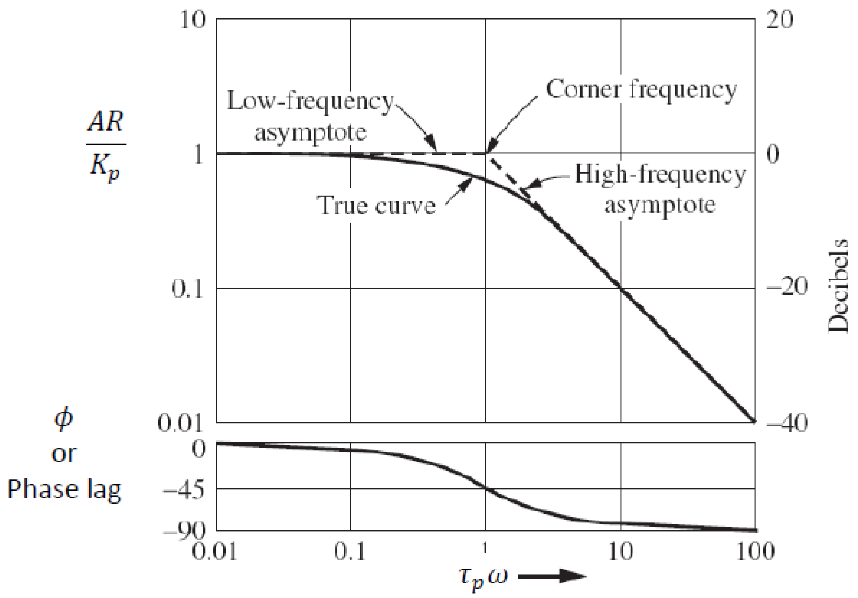


Figure: Bode diagram for first-order system.

Second –order system

$$AR = \frac{K_p}{\sqrt{(1-\tau^2\omega^2)^2 + (2\psi\tau\omega)^2}} \quad \phi = \tan^{-1}\left(\frac{-2\psi\tau\omega}{1-\tau^2\omega^2}\right)$$

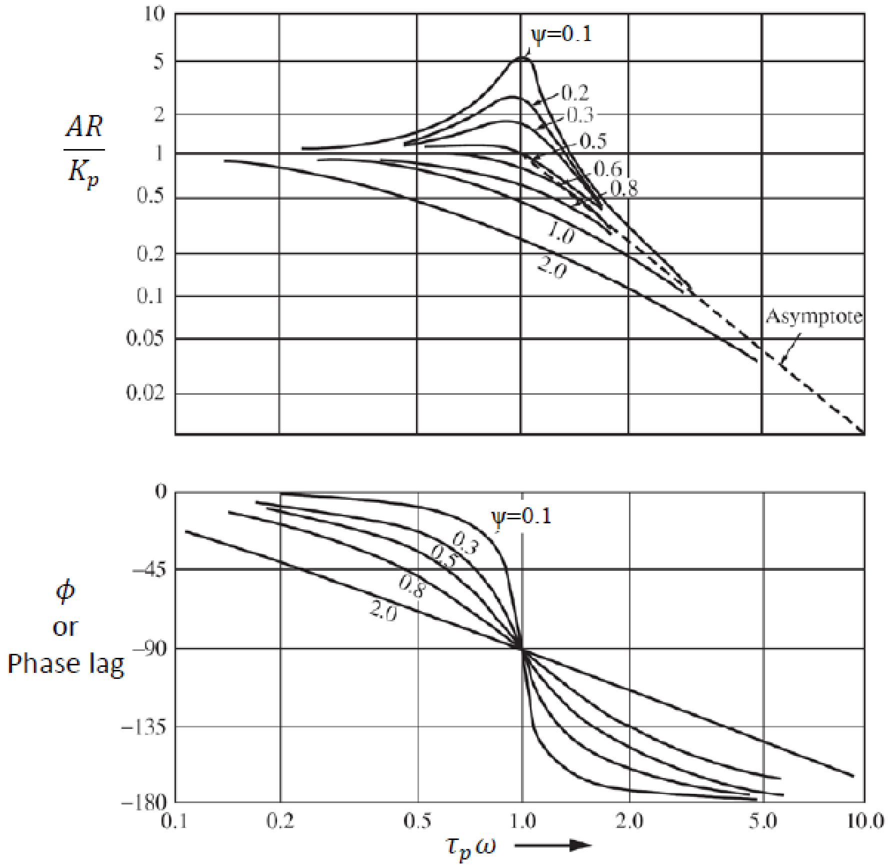


Figure: Block diagram for second-order system

$$\log\frac{AR}{K_p} = -\frac{1}{2}\log[(1-\tau^2\omega^2)^2 + (2\psi\tau\omega)^2]$$

1) as $\omega \to 0$, then $\log\frac{AR}{K_p} = -\frac{1}{2}\log(1) = 0$

$\therefore \frac{AR}{K_p} = 1$ stright line of a slope=0 **(L.F.A)**

$\phi = \tan^{-1}\frac{-0}{1} = 0$

2) as $\omega \to \infty$, then $\log\frac{AR}{K_p} = -\frac{1}{2}\log(\tau\omega)^4 = -2\log(\tau\omega)$ **(H.F.A)**

It is a straight line with a slope -2 passing through the pointAR=1 and τω=1

3) $\omega = \omega_c = \frac{1}{\tau}$

Pure dead-time system

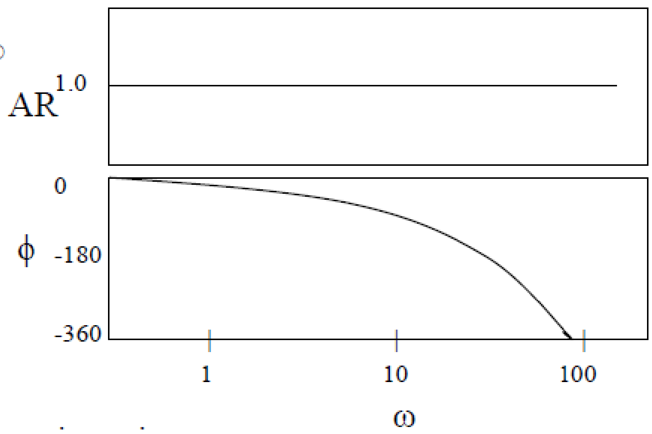
For the system

$AR = 1$

$\phi = -\tau_d\,\omega$

as $\omega \to 0$, $\phi = 0$

as $\omega \to \infty$, $\phi = \infty$



Example: Two systems in series

$$G_1(s) = \frac{1}{2s+1} \text{ and } G_2(s) = \frac{6}{5s+1}$$

The overall T.F. is

$$G(s) = \frac{1}{2s+1} \cdot \frac{6}{5s+1}$$

$$\therefore AR = \frac{1}{\sqrt{1+4\omega^2}} \cdot \frac{6}{\sqrt{1+25\omega^2}}$$

$$\log AR = \log 6 + \log(AR)_1 + \log(AR)_2$$

1- Region 1: From ω=0 to $\omega = \frac{1}{5}$, slope of the overall asymptote =0+0=0 (i.e. horizontal * going through the point AR=1)

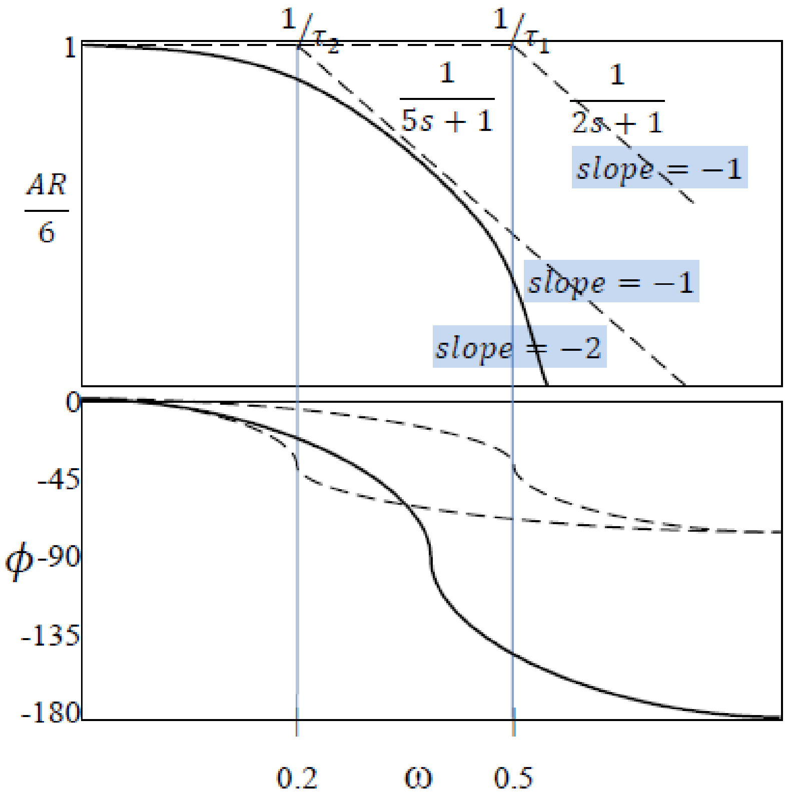
2- Region 2: From $\omega = \frac{1}{5}$ to $\omega = \frac{1}{2}$, slope of the overall asymptote =0+(-1)=-1 going through the point AR=1 , $\omega = \frac{1}{5}$

3- Region 3: From $\omega > \frac{1}{2}$, slope of the overall asyptote =(-1)+(-1)=-2

For ϕ

When as $\omega \to 0$, $\phi_1 \to 0$, $\phi_2 \to 0$, $\phi \to 0$

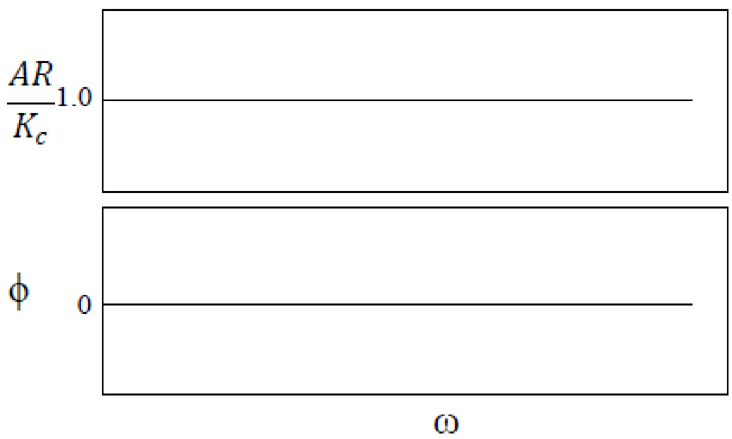
When as $\omega \to \infty$, $\phi_1 \to -90$, $\phi_2 \to -90$, $\phi \to -180$



Feedback Controller

1-Propertional controller

$AR = K_c \; \phi = 0$



2-Propertional Integral controller (PI)

$$AR = K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}}$$

$$\phi = \tan^{-1}(-\frac{1}{\omega\tau_I}) < 0$$

$$\log(\frac{AR}{K_c}) = \frac{1}{2}\log(1+\frac{1}{(\omega\tau_I)^2})$$

1- **Low frequency asymptote**

$$\text{as } \omega \to 0\,,\ \frac{1}{(\omega\tau_I)^2} >> 1 \quad \text{thenn} \log(\frac{AR}{K_c}) \to -\log(\omega\tau_I)$$

Consequently, the LFA is a straight line with slope=-1

$$\phi = \tan^{-1} - \frac{1}{0} = -90°$$

2- **High frequency asymptote**

$$\text{as } \omega \to \infty\,,\ \frac{1}{(\omega\tau_I)^2} \to 0 \quad \text{then} \to \log(\frac{AR}{K_c}) \to 0 \quad \text{i.e } \frac{AR}{K_c} \to 1$$

HFA is a horizontal line at value $\frac{AR}{K_c} = 1$

For the ϕ

as $\omega \to 0\,,\ \phi \to -90$

as $\omega \to \omega_c\,,\ \phi \to -45$

as $\omega \to \infty\,,\ \phi \to 0$

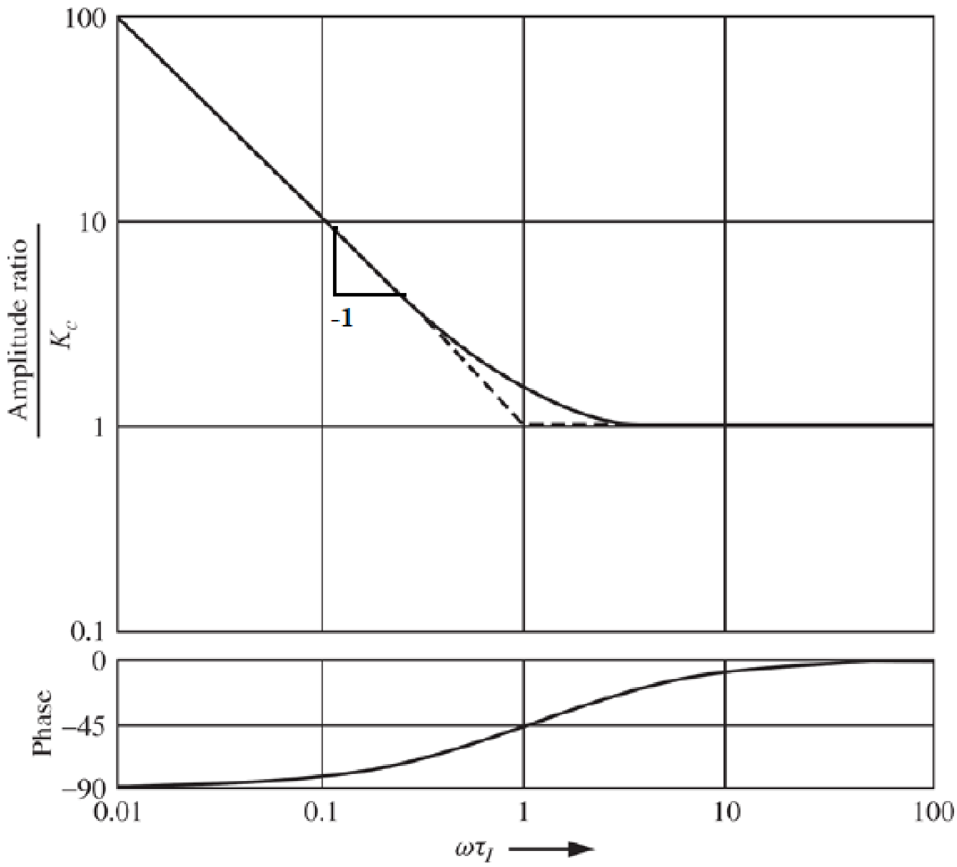


Figure Bode diagram for PI controller.

2-Propertional Derivative controller (PD)

$AR = K_c\sqrt{1+\tau_D^2\omega^2}$

$\phi = \tan^{-1}(\omega\tau_D) > 0$

1) **Low frequency asymptote**

$$\text{as } \omega \to 0 \text{ , } \log(\frac{AR}{K_c}) = \frac{1}{2}\log(\tau_D^2\omega^2) = 0 \Rightarrow \frac{AR}{K_c} = 1 \qquad \text{(L.F.A) slope=0}$$

$\phi = \tan^{-1}0 = 0^{\circ}$

2) **High frequency asymptote**

$$\text{as } \omega \to \infty \text{ , } \log(\frac{AR}{K_c}) = \frac{1}{2}\log(\tau_D^2\omega^2) = \log(\tau_D\omega) \qquad \text{(H.F.A) slope=+1}$$

$\phi = \tan^{-1}0 = 90^{\circ}$

as $\omega = 0$, $\phi = 0$

as $\omega = \omega_c$, $\phi = +45^{\circ}$

as $\omega = \infty$, $\phi \to +90^{\circ}$

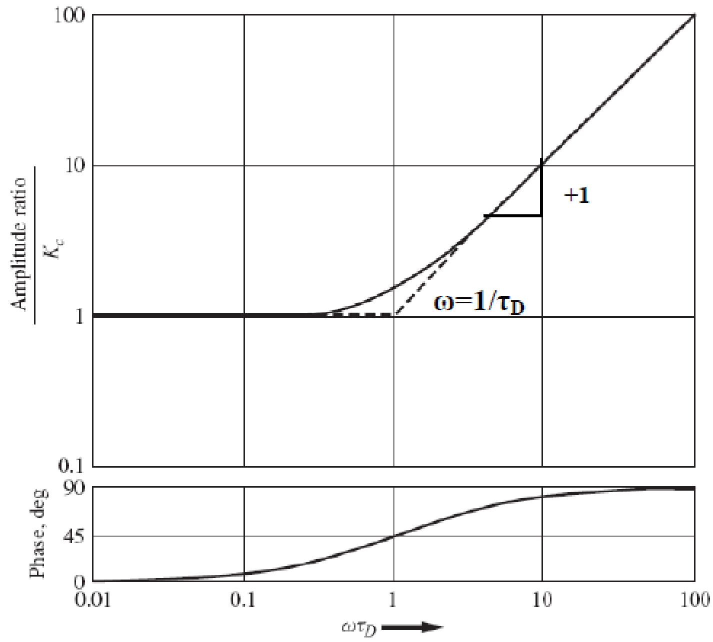


Figure: Bode diagram for PD controller.

3-Propertional Integral Derivative controller (PID)

$$\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega-\frac{1}{\tau_I\omega})^2}$$

$$\phi=\tan^{-1}(\tau_D\omega-\frac{1}{\tau_I\omega})$$

1) as $\omega\to 0$ then $\frac{AR}{Kc}=\sqrt{1+(\frac{1}{\tau_I\omega})^2}$ PI Controller

2) as $\omega\to\infty$ then $\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega)^2}$ PD Controller

3) as $\omega\to\frac{1}{\tau_I}$ then $\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega-1)^2}$

4) as $\omega\to\frac{1}{\tau_D}$ then $\frac{AR}{Kc}=\sqrt{1+(1-\frac{1}{\tau_I\omega})^2}$

<u>Frequency Response of non-interacting capactine in sereies</u>

$$G(s)=G_1(s)\times G_2(s)\times G_3(s)\times\ldots\ldots\ldots\times G_n(s)$$

$$\boxed{AR_t=AR_1\times AR_2\times AR_3\times\ldots\ldots\ldots\times AR_n}$$

$$\log AR_t = \log AR_1 + \log AR_2 + \log AR_3 + \ldots\ldots\ldots\ldots + \log AR_n$$

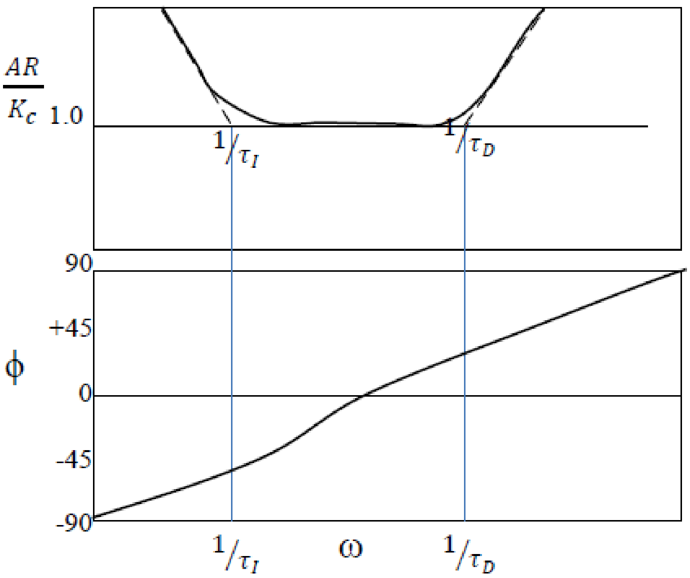
$$\phi_t = \phi_1 + \phi_2 + \phi_3 + \ldots\ldots\ldots\ldots + \phi_n$$

Example: Bode Digram of PID Controller

$$G_1(s) = 10(1 + \frac{1}{10s} + 5s)$$

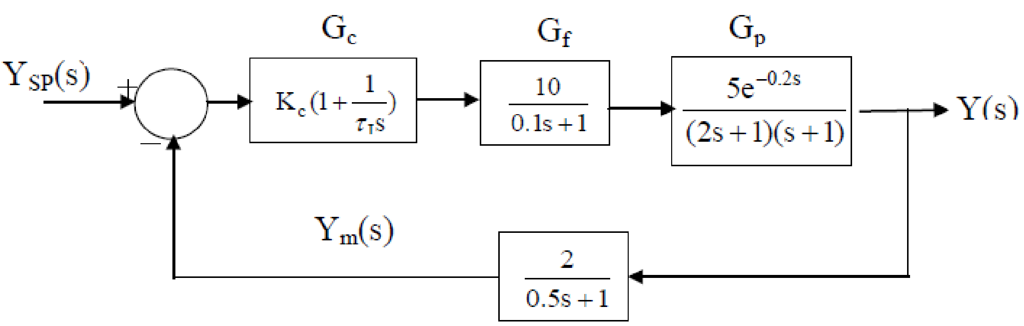
$$\omega_{c1}(s) = \frac{1}{10} = 0.1 \qquad \text{signal}(-1)$$

$$\omega_{c1}(s) = \frac{1}{5} = 0.2 \qquad \text{signal}(+1)$$



Example:

Bode plots of open loop system



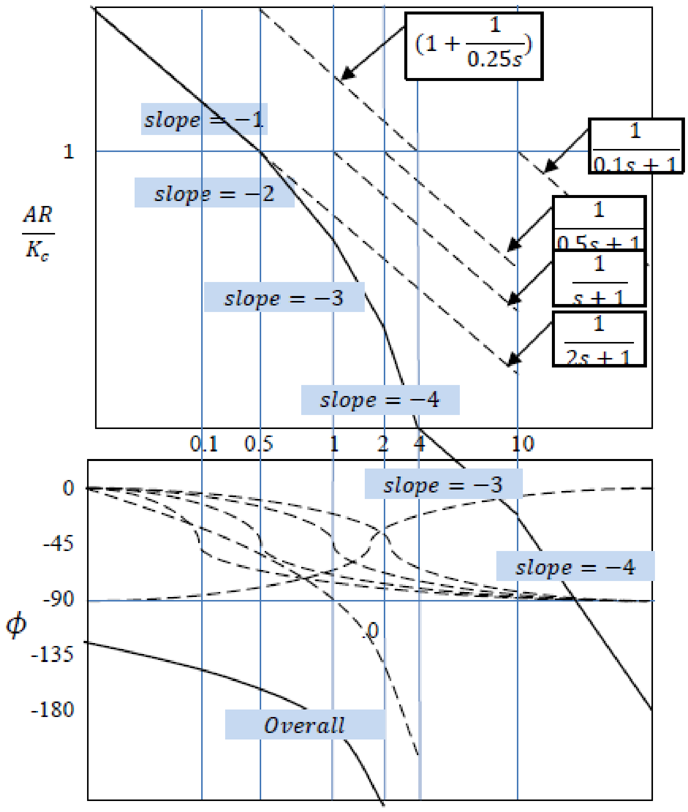
The Open loop T.F. of the feedback control

$$G_{OL} = G_c\, G_f\, G_p\, G_m$$

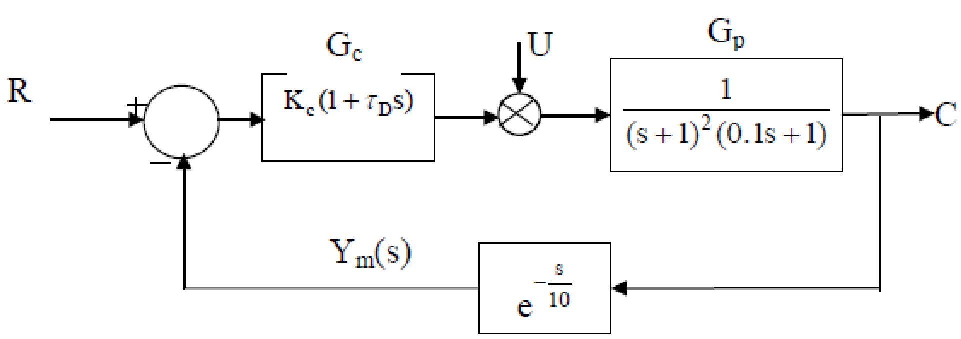
$$G_{OL} = 100K_c(1 + \frac{1}{\tau_I s})\frac{1}{0.1s+1}\cdot\frac{1}{(2s+1)(s+1)}\cdot\frac{1}{(0.5s+1)}e^{-0.2s}$$

With K_c=4 and τ_I=0.25

$\therefore$K=400



Example: Plot the B.D. for the open loop T.F. for the fig. below



For Kc=10 and τ_D=0.5 the overall transfer function is

$$G_{OL}(s) = \frac{10(0.5s+1)e^{-\frac{s}{10}}}{(s+1)^2(0.1s+1)}$$

Overall Bode diagram

$G_1(s) = \frac{1}{1s+1} \rightarrow \omega_{c1} = \frac{1}{1} = 1$ **stright line slope=-1**

$G_2(s) = \frac{1}{1s+1} \rightarrow \omega_{c2} = \frac{1}{1} = 1$ **stright line slope=-1**

$G_3(s) = 0.5s+1 \rightarrow \omega_{c3} = \frac{1}{0.5} = 2$ **stright line slope=+1**

$G_4(s) = \frac{1}{0.1s+1} \rightarrow \omega_{c4} = \frac{1}{0.1} = 10$ **stright line slope=-1**

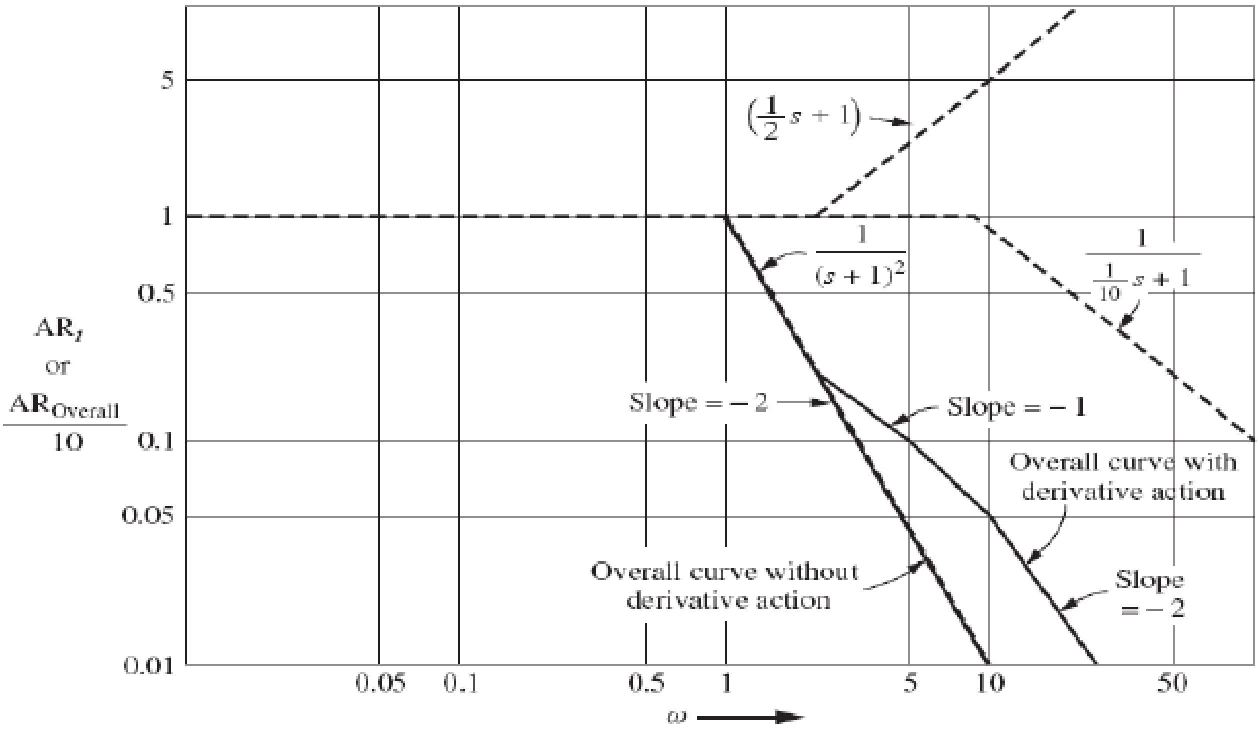
$G_5(s) = e^{-\frac{s}{10}}$ **stright line slope=0**

Amplitude Ratio Curve Prediction

ω	SL1	SL2	SLC	SL3	sLd	SLTotal
0-1	0	0	0	0	0	0
1-2	-1	-1	0	0	0	-2
2-10	-1	-1	+1	0	0	-1
10-	-1	-1	+1	-1	0	-2

Phase Lag Curve Prediction

ω	ϕ L1	ϕL2	ϕ C	ϕ L3	ϕ d	SLTotal
0	0	0	0	0	-	-
1	-45	-45	0	-	-	-
2	-	-	45	-	-	-
10	-	-	-	-45	-	-
∞	-90	-90	90	-90	-∞	-∞



(a)

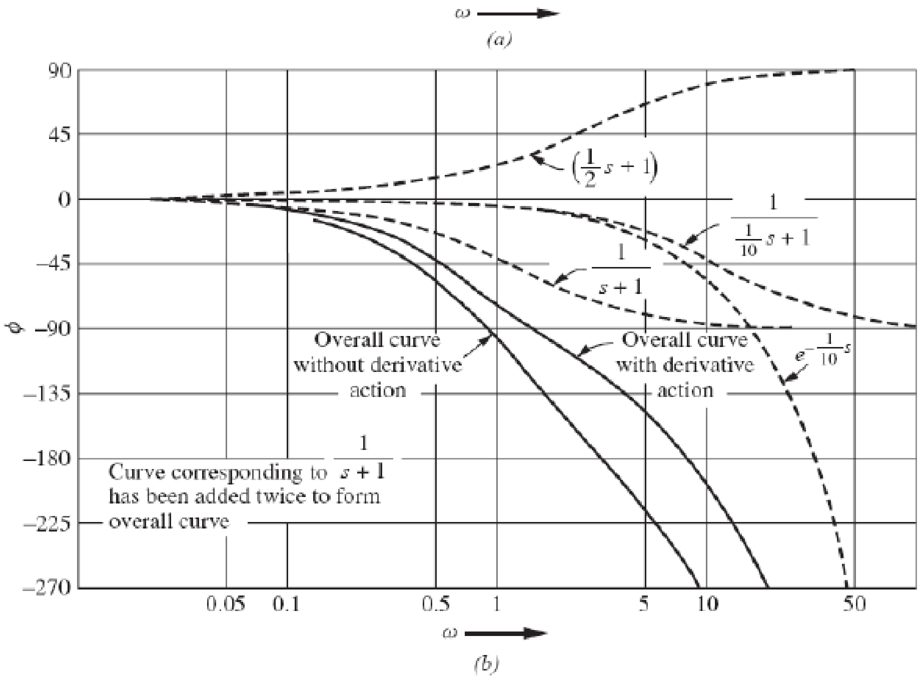
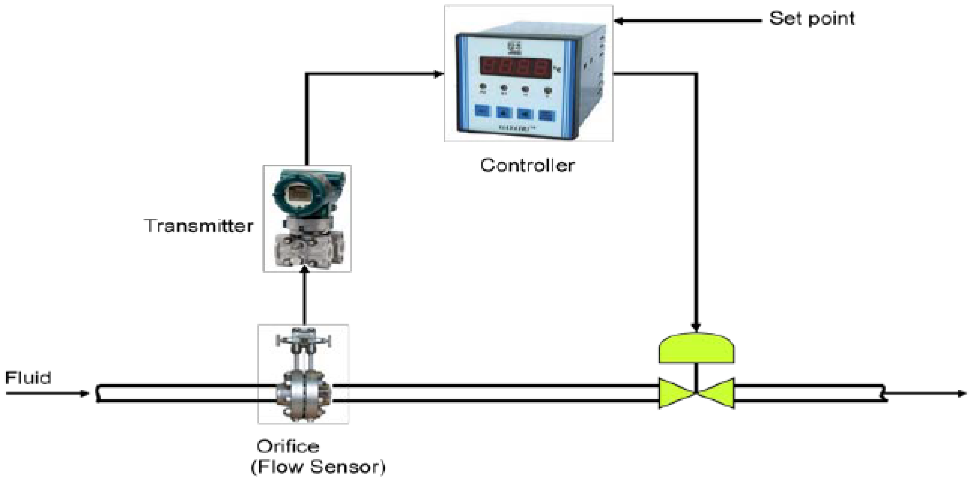


Figure: Block diagram for: (*a*) Amplitude ratio; (*b*) phase angle.

The Instrumentation and Control Diagrams



Instrumentation

The example level-control problem had three critical pieces of instrumentation: a sensor (measurement device), actuator (manipulated input device), and controller. The sensor measured the tank level, the actuator changed the flow rate, and the controller determined how much to vary the actuator, based on the sensor signal. Each device in a control loop must supply or receive a signal from another device.

Sensors (Sensing Element)

A device, usually electronic, which detects a variable quantity and measures and converts the measurement into a signal to be recorded elsewhere. A sensor is a device that measures a physical quantity and converts it into a signal which can be read by an observer or by an instrument.

There are many common sensors used for chemical processes. These include temperature, level, pressure, flow, composition, and pH.

For example, a mercury thermometer converts the measured temperature into expansion and contraction of a liquid which can be read on a calibrated glass tube. A thermocouple converts temperature to an output voltage which can be read by a voltmeter.

Control of unit operations

1) **Level Control**

- A level control is needed whenever there is a V/L or L/L interface
- Many smaller vessels are sized based on level control response time

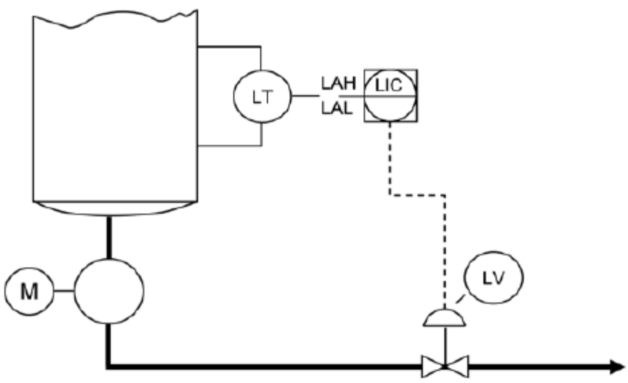


Figure 14.1 Liquid level control system

Example: A boiler drum with a conventional feedback control system is shown in Fig. 14.2. The level of the boiling liquid is measured and used to adjust the feed water flow rate.

This control system tends to be quite sensitive to rapid changes in the disturbance variable, steam flow rate, as a result of the small liquid capacity of the boiler drum.

Rapid disturbance changes can occur as a result of steam demands made by downstream processing units.

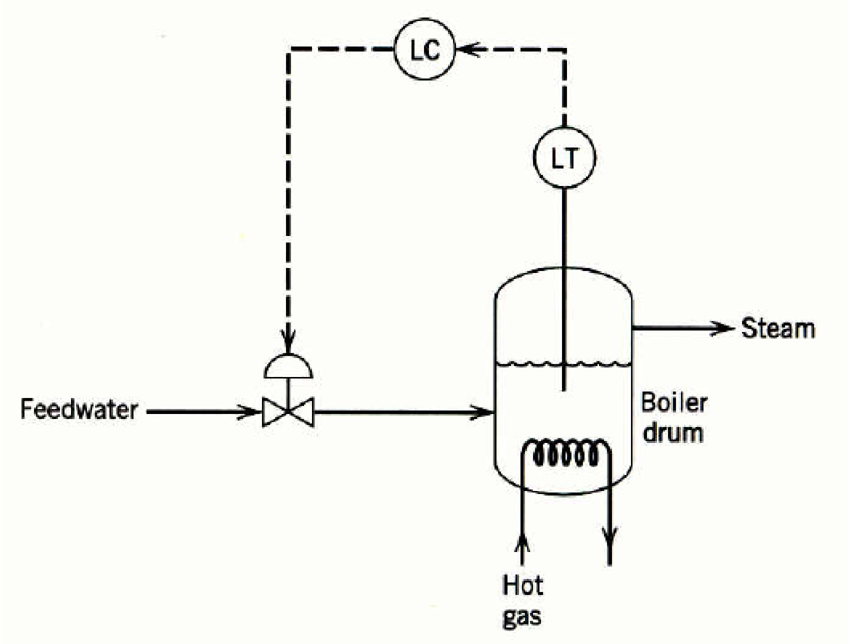


Figure 14.2 The feedback control of the liquid level in a boiler drum.

The feedforward control scheme in Fig. 14.3 can provide better control of the liquid level. Here the steam flow rate is measured, and the feedforward controller adjusts the feedwater flow rate.

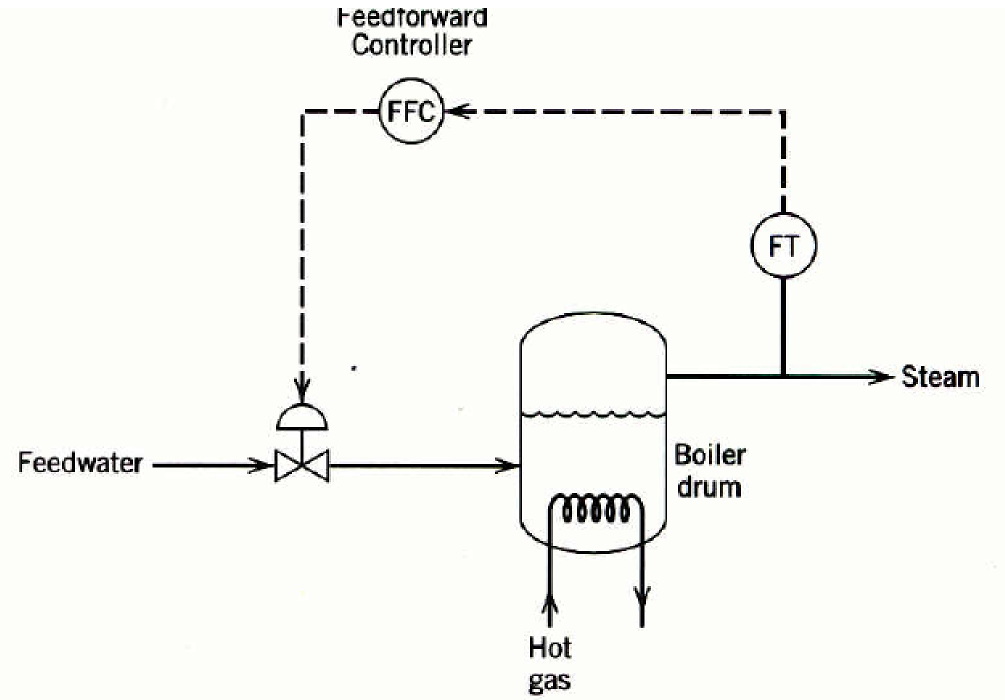


Figure 14.3 The feedforward control of the liquid level in a boiler drum.

2) Pressure Control

- Pressure control is usually by venting a gas or vapor.
- In hydrocarbon processes, off-gas is often vented to fuel.

- In other processes, nitrogen may be brought in to maintain pressure and vented via scrubbers.
- Most common arrangement is direct venting.
- Several vessels that are connected together may have a single pressure controller.

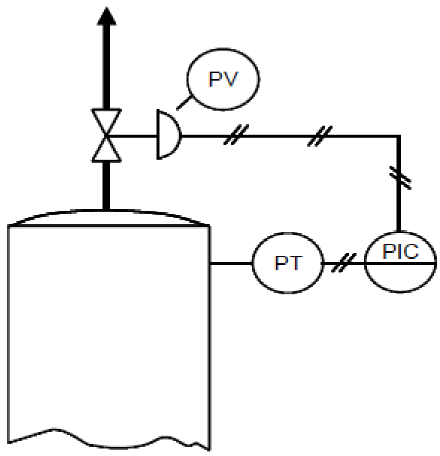


Figure 14.4 Pressure control system

3) Flow Control

- Most common arrangement is a control valve downstream of a pump or compressor.

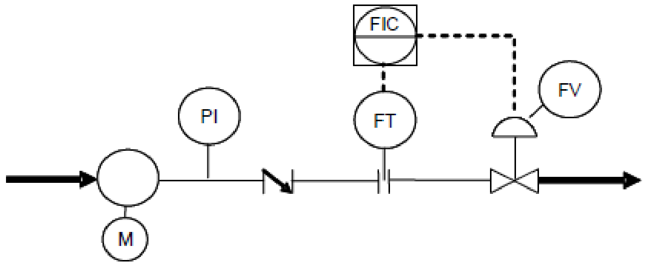


Figure 14.5 Flowrate control system

Example:Vaporizer Flow Control

- Vaporizer flow control needs to prevent liquid accumulation.
- Hence use level controller to actuate heat input to the vaporizer and maintain a constant inventory.
- Control of liquid flow in is easier than control of vapor flow out.

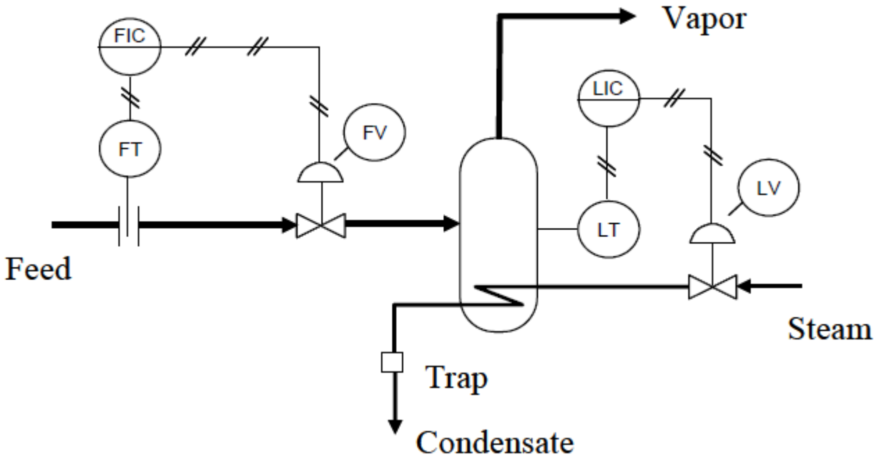


Figure 14.6 Vaporizer control system

4) Temperature Control: Single Stream

- Heaters and coolers are usually controlled by manipulating the flow rate of the hot or cold utility stream.
- Final control element can be on inlet or outlet of utility side.

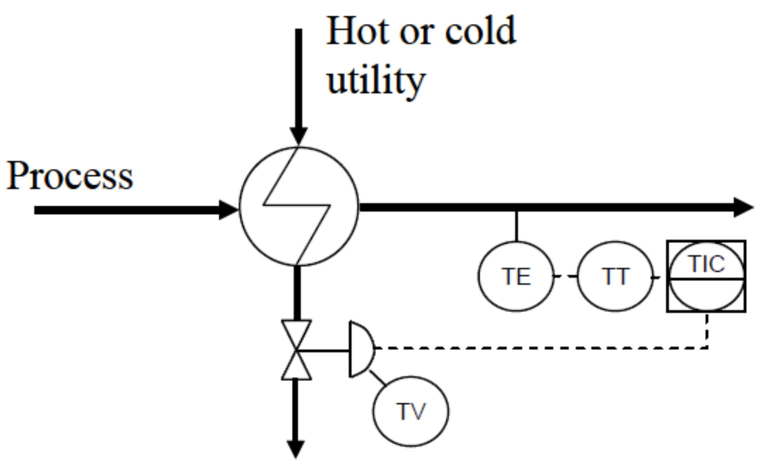


Figure 14.7 Temperature control system

Example: Heat exchangers temperature control

- Temperature control for an heat exchanger is usually by manipulating the flow through a bypass.
- Only one side of an exchanger can be temperature controlled.
- It is also common to see heat exchangers with temperature control on the downstream heater and cooler.

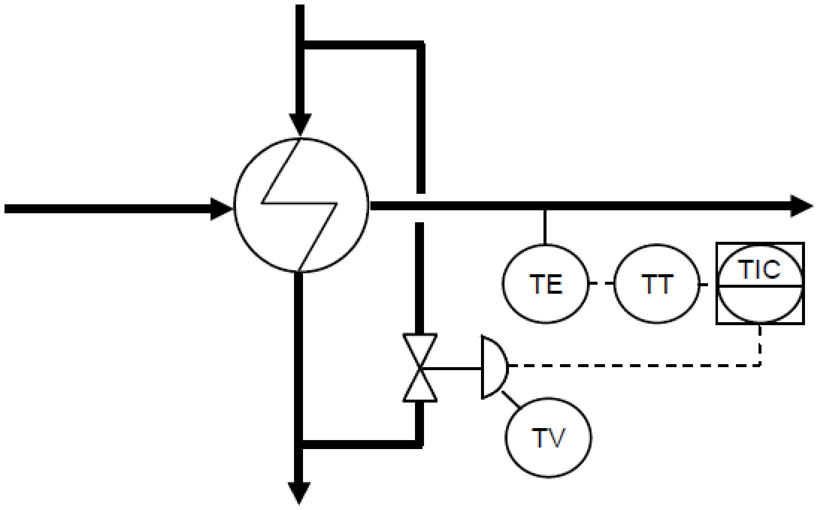


Figure 14.8 Temperature control of heat exchanger

Example: Air coolers temperature control

- Ambient air temperature varies, so air coolers are oversized and controlled by manipulating a bypass.
- Alternatively, air cooler can use a variable speed motor, louvers or variable pitch fans.

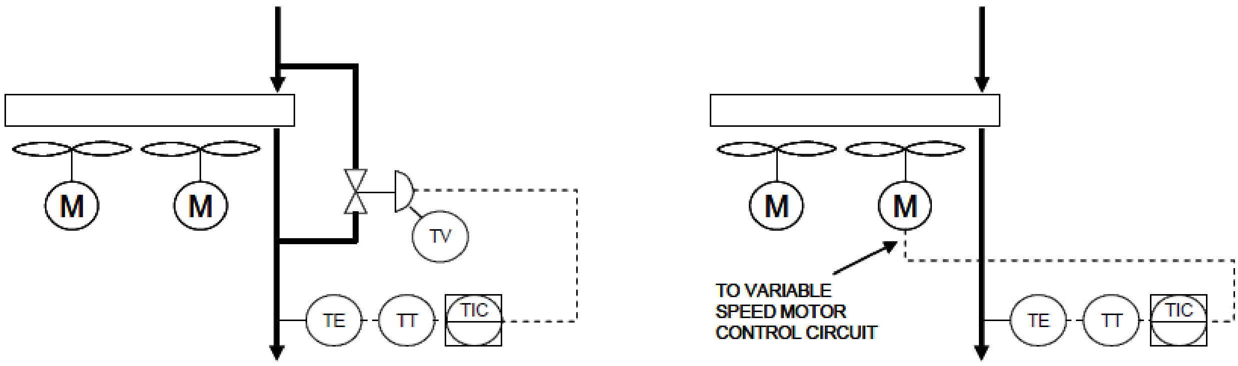


Figure 14.9 Temperature control of air coolers

Example: Temperature Control of CSTR

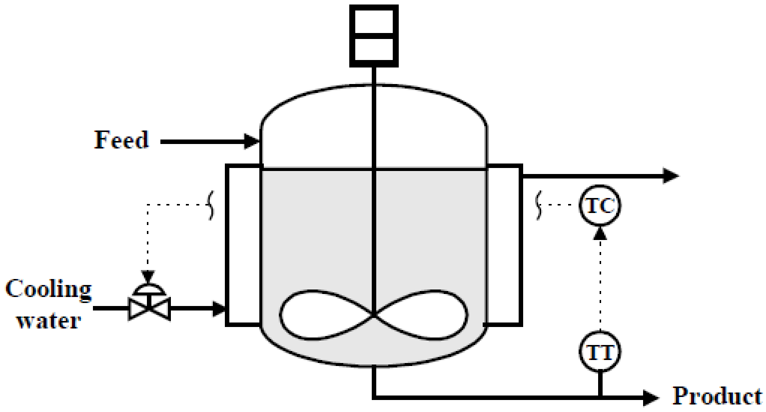


Figure 14.10 Temperature control of CSTR

Distillation Control

- ❖ Distillation control is a specialized subject in its own right.
- ❖ In addition to controlling condenser pressure and level in the sump, a simple distillation column has two degrees of freedom.
  - ▪ Material balance (split) and energy balance (heat input or removed).
  - ▪ Therefore needs two controllers.

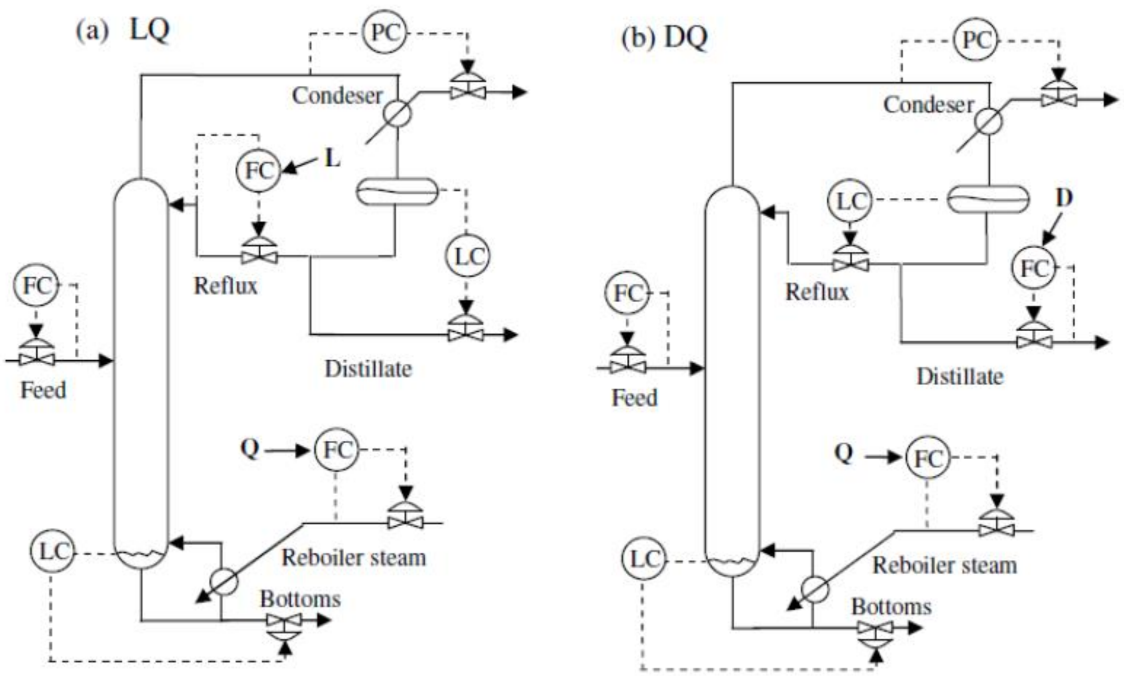
- ▪ Therefore has the possibility that the controllers will interact and "fight" each other.
- ❖ Side streams, intermediate condensers & reboilers, pump-arounds, etc. all add extra complexity and degrees of freedom.

The Energy Balance (LQ) Distillation Column Control Structure

The LQ control structure is the most natural control structure for a simple distillation column. This is because the separation in a distillation column occurs due to successive condensation and vaporization of the counter-current vapour and liquid streams flowing through the column. Adjusting the cold reflux, the source of condensation, and the reboiler duty, the source of vaporization, is then a natural choice for regulating the separation achieved in the column. The LQ control structure shown in figure (14.11 a) is thus the most commonly applied distillation control structure. It is also sometimes referred to as an energy balance structure as changing L (cold reflux) or Q alters the energy balance across the column to affect the distillate to bottoms product split.

Material Balance Distillation Column Control Structures

The other control structures are referred to as material balance structures as the product split is directly adjusted by changing the distillate or bottoms stream flow rate. The material balance structures are applied when a level loop for the LQ structure would be ineffective due to a very small product stream (D or B) flow rate. Figure 14.11 b, c and D show Schematics of DQ, LB and DB distillation column control structures. The DQ structure is thus appropriate for columns with very large reflux ratio (L/D > 4). The distillate stream flow is then a fraction of the reflux stream so that the reflux drum level cannot be maintained using the distillate. The level must then be controlled using the reflux. The LB structure is appropriate for columns with a small bottoms flow rate compared to the boil-up. The bottoms stream is then not appropriate for level control and the reboiler duty must be used instead. The DB control structure is used very rarely as both D and B cannot be set independently due to the steady state overall material balance constraint. In dynamics however, the control structure may be used when the reflux and reboil are much larger than the distillate and bottoms respectively.



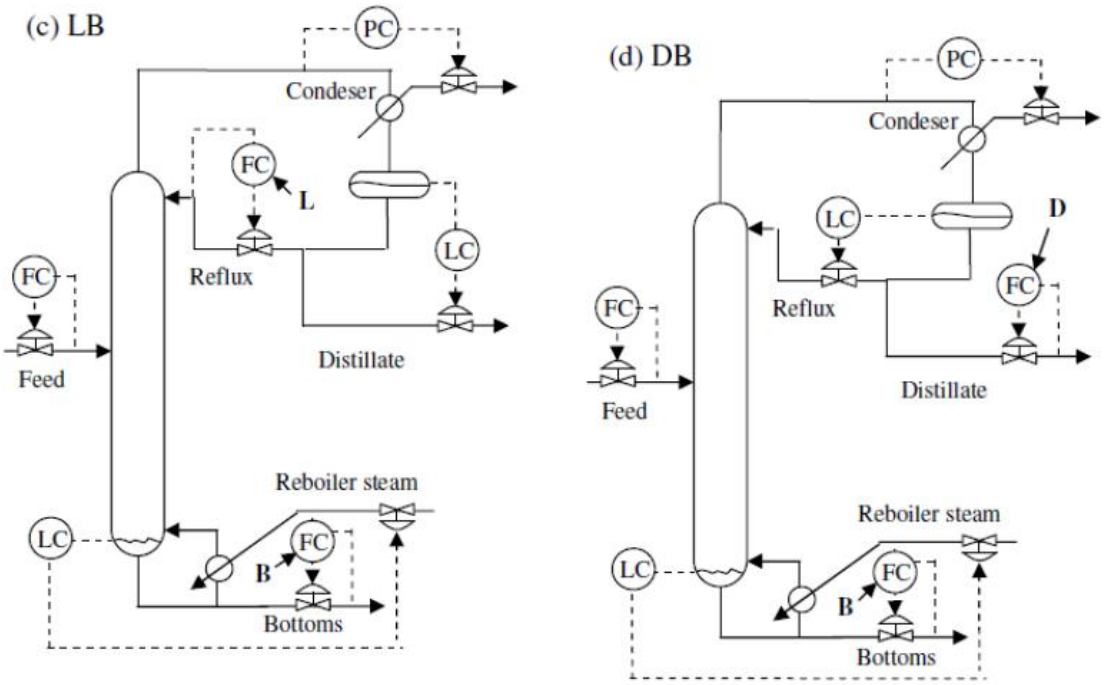


Figure 14.11 Schematics of LQ, DQ, LB and DB distillation column control structures

Other Distillation Column Control Structure

Other variants of the basic control structure types include the L/D-Q, L/D-B and DQ/B. In the first two structures the reflux ratio is adjusted for regulating the separation. In the last structure the reboil ratio is adjusted. These control structures are illustrated in Figure 14.12.

Note that when the reflux is adjusted in ratio with the distillate, the distillate stream can be used to control the reflux drum level even as it may be a trickle compared to the reflux rate.

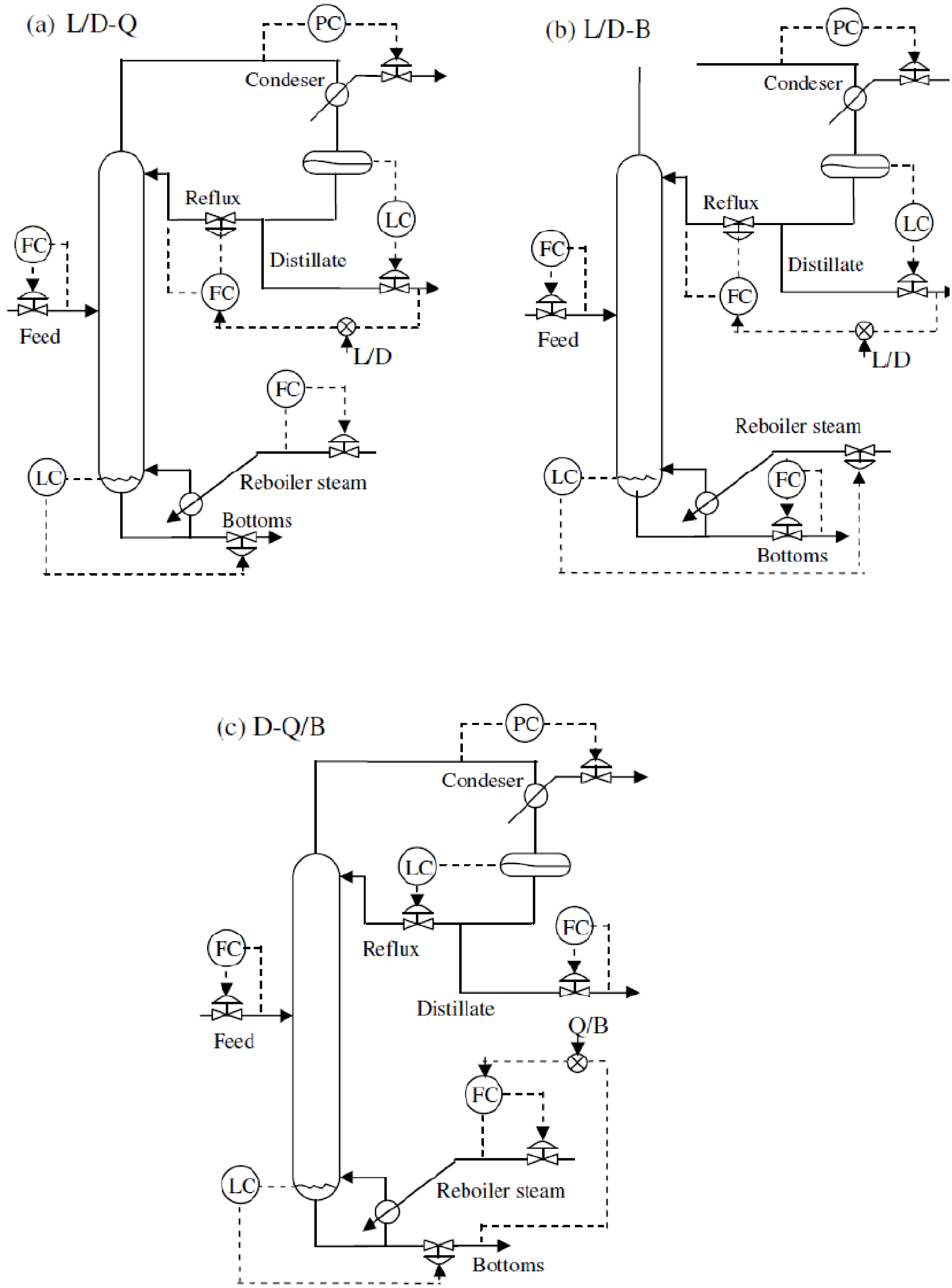


Figure 14.12 Schematics of L/D-Q, L/D-B, and D-Q/B distillation column control structures.

Batch Distillation

- Reflux flow control adjusted based on temperature (used to infer composition)

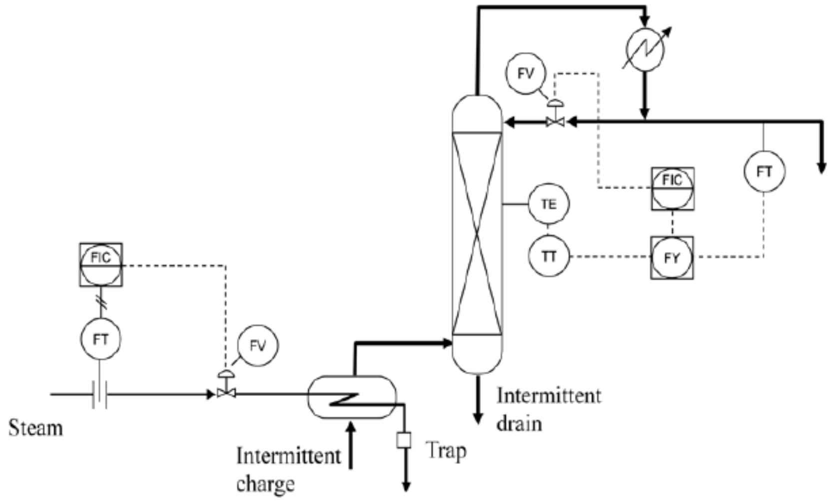
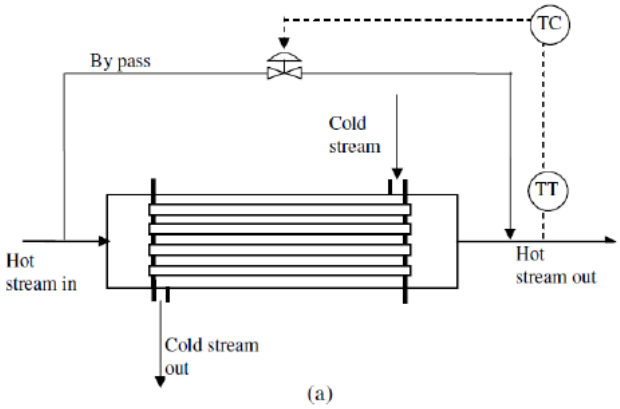


Figure 14.13 Batch distillation column control system

Heat Exchangers

Heat exchangers process used to transfer heat between two process streams. The flow of these process streams is usually set elsewhere in the plant so that adjusting the flowrate of one of the process streams to regulate the amount of heat transferred is not possible.

To provide a control degree-of-freedom for regulating the heat transferred, a small by-pass (~5-10%) of one of the process streams around the heat exchanger is provided. The outlet temperature of this process stream or the other process stream can be controlled by manipulating the by-pass rate. These two schemes are illustrated in Figure 14.14. In the former, tight temperature control is possible as the amount of heat transferred is governed by the bypass. In the latter, a thermal lag of the order of 0.5 to 2 minutes exists between the manipulated and controlled variable.



(a)

Figure 14.14 By-pass control of process to process heat exchangers
(a) Controlling and bypassing hot stream (b) Controlling cold stream and bypassing hot stream

Control of Miscellaneous Systems

Vapor Absorption Cycle

In addition to compression systems, refrigerant absorption systems are also applied industrially. The absorption based refrigeration cycle and its control scheme is shown in Figure 14.15. Ammonia (refrigerant) rich strong liquor is distilled at high pressure to recover liquid ammonia as the distillate and ammonia lean weak liquor as the bottoms. The liquid ammonia is fed to the evaporator where it absorbs heat from the process stream to be chilled and evaporates. Vapor ammonia is absorbed by the 'weak liquor' water stream. The 'strong liquor' so formed is fed to the distillation column to completed the closed circuit refrigerant loop. The temperature of the chilled process stream is controlled by adjusting the level setpoint of the evaporator. The heat transfer rate is thus varied by changing the area across which heat transfer occurs. The evaporator level controller adjusts the distillate liquid ammonia flow. An increase in the level of the evaporator implies an increase in the ammonia evaporation rate so that the weak liquor rate is increased in ratio to absorb the ammonia vapours. The strong liquor is cooled and collected in a surge drum. The level of the surge drum is not controlled. Liquid from the surge drum is pumped back to the distillation column through a process-to-process heater that recovers heat from the hot 'weak liquor' bottoms from the distillation column. The flow rate of the strong liquor to the column is adjusted to maintain the column bottoms level. Also, the steam to the reboiler is manipulated to maintain a tray temperature.

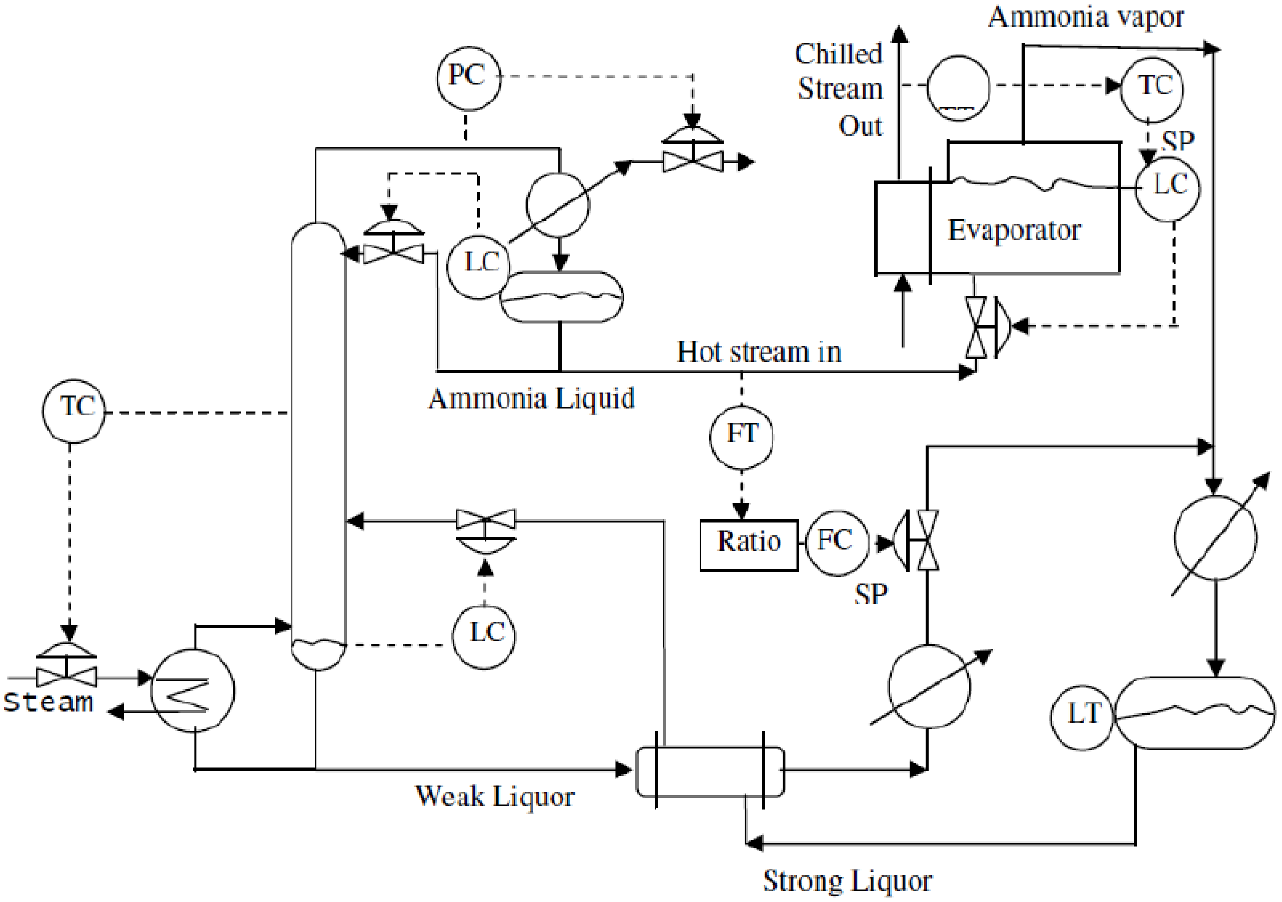


Figure 14.15 Absorption refrigeration control system